Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017
Mathematics 2 (Curriculum Economics, Markets and Finance)

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Schematic solution of the exercises

Code A

Exercise 1. *Consider the vectors*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 4 \\ k \end{pmatrix},$$

where k is a real number.

a) *Find for which values of k the vectors are linearly independent.*
b) *Set $k = -3$. Write $\vec{v}_3$ as a linear combination of the others.*
c) *Set $k = 1$. Find the inverse of the matrix whose columns are the given vectors and check that the product of the matrix by its inverse is the identity matrix.*

Solution. The determinant of the matrix whose columns are the given vectors is $-4k-12$. So if $k=-3$ the vectors are linearly dependent, if $k \neq -3$ they are linearly independent.

If $k=-3$ the vectors are linearly dependent, as already observed. So at least one of the three is a linear combination of the others. To find whether $\vec{v}_3$ is a linear combination of the others we must solve the following equation

$$\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2,$$

that is the linear system

$$\begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 = 4 \\ -c_1 + c_2 = -3 \end{cases}.$$

This system has only the solution $c_1 = 2$, $c_2 = -1$.

If $k = 1$ we obtain the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 4 \\ -1 & 1 & 1 \end{pmatrix}.$$

The inverse is

$$A^{-1} = \frac{1}{16}\begin{pmatrix} 4 & 2 & -8 \\ 6 & -1 & 4 \\ -2 & 3 & 4 \end{pmatrix}.$$

Computing the product AA^{-1} is a straightforward calculation. □

Exercise 2. *Consider the two variables real function*

$$f(x,y) = x^2 + y^3 - 4x - 12y + 1.$$

a) *Find all local maximum, minimum and saddle points. In case of maxima or minima find also the corresponding values of the function.*
b) *Consider the constraint $y = x$. Find all local maximum and minimum points of the function f on this constraint, without the use of Lagrangian multipliers. Say whether the function has global maximum or minimum on the constraint.*

Solution. The first order necessary conditions are

$$\begin{cases} f'_x = 2x - 4 = 0 \\ f'_y = 3y^2 - 12 = 0 \end{cases} \quad \Rightarrow \quad \mathsf{A} = (2,2), \quad \mathsf{B} = (2,-2).$$

Now we use the second order sufficient conditions.

$$\begin{cases} f''_{xx} = 2 \\ f''_{yy} = 6y \\ f''_{xy} = f''_{yx} = 0 \end{cases} \quad \Rightarrow \quad H(\mathsf{A}) = 24 > 0 \quad \text{and} \quad f''_{xx}(\mathsf{A}) = 2 > 0, \quad H(\mathsf{B}) = -24.$$

So A is a minimum point and B a saddle point. The value of the minimum is -19.

If $y = x$ we obtain the one variable function $g(x) = x^3 + x^2 - 16x$, whose derivative is $g'(x) = 3x^2 + 2x - 16$. So this function has a local maximum at $x = -8/3$ and a local minimum at $x = 2$. As $y = x$ the corresponding points on the cartesian plane are $(-8/3, -8/3)$ and $(2,2)$. As

$$\lim_{x \to \pm\infty} g(x) = \pm\infty,$$

there is no absolute maximum or minimum. □

Exercise 3. *Consider the two variables real function*

$$f(x,y) = x^2 + y^2 + 2y$$

and the constraint $g(x,y) = x^2 + 4y^2 - 1 = 0$. Suppose you know that the constraint is an ellipse (so it is closed and bounded).

Find the global maximum and minimum of the function on the constraint using the Lagrangian multipliers method.

Solution. The Lagrangian function is

$$\mathcal{L} = x^2 + y^2 + 2y - \lambda(x^2 + 4y^2 - 1).$$

The necessary first order conditions are

$$\begin{cases} \mathcal{L}'_x = 2x - 2\lambda x = 0 \\ \mathcal{L}'_y = 2y + 2 - 8\lambda y = 0 \\ x^2 + 4y^2 - 1 = 0 \end{cases}.$$

From the first equation we obtain $x = 0$ or $\lambda = 1$. If $x = 0$ from the third equation we obtain $y = \pm 1/2$, and from the second $\lambda = 3/4$ and $\lambda = -1/4$. If $\lambda = 1$, from the second equation we obtain $y = 1/3$ and form the third $x = \pm\sqrt{5/9}$.

So we have four points to check. The corresponding values of the function are:

$$\frac{5}{4}, \quad -\frac{3}{4}, \quad \frac{4}{3}, \quad \frac{4}{3}.$$

So the global maximum is $4/3$ and the global minimum is $-3/4$. □

Code B

The exercises are almost the same as with code A: only limited changes in some coefficients occur. The solution strategies are exactly the same. So we only give the detailed results of the calculations.

Exercise 1. *Consider the vectors*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 3 \\ 3 \\ k \end{pmatrix},$$

where k is a real number.

a) *Find for which values of k the vectors are linearly independent.*
b) *Set $k = -2$. Write $\vec{v}_3$ as a linear combination of the others.*
c) *Set $k = 1$. Find the inverse of the matrix whose columns are the given vectors and check that the product of the matrix by its inverse is the identity matrix.*

Solution. The determinant of the matrix whose columns are the given vectors is $6 + 3k$. So if $k = -2$ the vectors are linearly dependent, if $k \neq -2$ they are linearly independent.

If $k = -2$ the vectors are linearly dependent, as already observed. So at least one of the three is a linear combination of the others. To find whether $\vec{v}_3$ is a linear combination of the others we must solve the following equation

$$\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2.$$

Proceeding as in code A we obtain only the solution $c_1 = 2$, $c_2 = -1$.

The requested inverse matrix is

$$A^{-1} = \frac{1}{9}\begin{pmatrix} -5 & 7 & -6 \\ -2 & 1 & 3 \\ 4 & -2 & 3 \end{pmatrix}.$$ □

Exercise 2. *Consider the two variables real function*

$$f(x, y) = x^3 + y^2 - 12x - 4y + 2.$$

a) *Find all local maximum, minimum and saddle points. In case of maxima or minima find also the corresponding values of the function.*
b) *Consider the constraint $y = x$. Find all local maximum and minimum points of the function f on this constraint, without the use of Lagrangian multipliers. Say whether the function has global maximum or minimum on the constraint.*

Solution. Proceeding exactly as with code A we obtain the points $\mathsf{A} = (2, 2)$, that is a minimum and $\mathsf{B} = (-2, 2)$, that is a saddle. The value of the minimum is -18.

As regards the constrained optimization problem calculations and results are the same as with code A. □

Exercise 3. *Consider the two variables real function*

$$f(x, y) = x^2 + y^2 + 2x$$

and the constraint $g(x, y) = 4x^2 + y^2 - 1 = 0$*. Suppose you know that the constraint is an ellipse (so it is closed and bounded).*

Find the global maximum and minimum of the function on the constraint, using the Lagrangian multipliers method.

Solution. The exercise is almost exactly identical to the corresponding exercise of code A: only change x with y and viceversa. The global max and min are the same. □