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Precalculus

A Prelude to Calculus with Exercises

Notes for a crash-course in Mathematics, Università di Bologna,
Corso di Laurea in Economia e Commercio, Curriculum Management - Forlì

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The power of mathematics is often to change one thing into another, to change geometry into language.

Marcus du Sautoy

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A Word from the Authors

In this work you will find all the contents of a crash-course in Mathematics, intended for students at the first year of a University Course in Economics.

If you find any mistake, please let us know at the mail address batmath@gmail.com.

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1 Basic concepts

The aim of this chapter is to recall some basic mathematical concepts that will be needed in later chapters: special products and factors, radicals and fractions.

1.1 Special products and factors

Some mathematical problems require the computation of products involving two or more variables to simplify expressions and to obtain the desired results. Many of these products are *special* because they are very common, and they are worth knowing. If we are able to recognize these products easily, it makes our life easier later on. On the other hand, it is also important to be able to write an algebraic sum as the product of its simplest factors, i.e. performing *factorization* or *factoring*.

In what follows, some (of the most common) examples of special products and factors are presented.

1.1.1 Factoring an algebraic expression

Consider the following examples.

Example 1.1. $6x + 2x^2y + 4xy^2 = 2x(3 + xy + 2y^2)$. This means that the *factors* of $6x + 2x^2y + 4xy^2$ are the monomial $2x$ and the polynomial $3 + xy + 2y^2$, and that the *common factor* is $2x$.

Example 1.2. $a^2b + ab^2 = ab(a + b)$.

Example 1.3. $(a + b)^2 - 2b(a + b) + 2a(a + b) = (a + b)(a + b - 2b + 2a) = (a + b)(3a - b)$. Here the common factor $(a + b)$ is a polynomial and not a monomial as in previous Examples 1.1 and 1.2.

Example 1.4. $3b^2(x^2 + y) - 6b^3(x^2 + y) + 12b^4(x^2 + y) = 3b^2(x^2 + y)(1 - 2b + 4b^2)$. The common factors are the monomial $3b^2$ and the polynomial $(x^2 + y)$.

Sometimes, factoring is more hard-working

Example 1.5. $ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b)$,

and, sometimes, more creative

Example 1.6. $ax - bx + by - ay - b + a = x(a - b) - y(a - b) + (a - b) = (a - b)(x - y + 1)$.

In algebra, multiplying binomials is easier if we are able to recognize their patterns. We multiply sum and difference of binomials and multiply by squaring and cubing to find some of the special products.

1.1.2 Product of a sum and a difference

If we multiply the sum and the difference of two quantities a and b we get the following rule

$$(a + b)(a - b) = a^2 - b^2 . \tag{1.1}$$

So, the product of a sum and a difference of the same two terms is equal to the difference of the squares of the terms. If we read Equation (1.1) from right to left we see that the difference of squares of two terms is equal to the product of the sum and the difference of the same two terms.
The quantities a and b can be any two monomials and polynomials.

Note 1.1. Note that *it is not* possible to recognize a pattern for the sum of squares of two terms a^2+b^2.

Consider now the following examples.

Example 1.7. $(x-1)(x+1)=x^2-1$.

Example 1.8. $(-x-2)(-x+2)=(-x)^2-4=x^2-4$.

Example 1.9. $x^2-3=(x-\sqrt{3})(x+\sqrt{3})$.

Example 1.10. $(xy+2x+3y)(xy+2x-3y)=[(xy+2x)+3y][(xy+2x)-3y]=(xy+2x)^2-(3y)^2=x^2y^2+4x^2y+4x^2-9y^2$.

1.1.3 Square of a binomial

To compute $(a+b)^2$, i.e. square of a binomial, we have the following rule

$$(a+b)^2=a^2+2ab+b^2. \tag{1.2}$$

So, to square $(a+b)$, we square the first term (a^2), add twice the product of the two terms $(2ab)$, then add the square of the last term (b^2). Similarly, to compute $(a-b)^2$, we have the following rule

$$(a-b)^2=a^2-2ab+b^2. \tag{1.3}$$

So, to square $(a-b)$, we square the first term (a^2), subtract twice the product of the two terms $(-2ab)$, then add the square of the last term (b^2).
The quantities a and b can be any two monomials and polynomials. Equations (1.2) and (1.3) permit also to factor the special trinomials $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

Consider the following examples.

Example 1.11. $(x+2y)^2=x^2+2\cdot x\cdot 2y+(2y)^2=x^2+4xy+4y^2$.

Example 1.12. $(4x-3y)^2=(4x)^2-2(4x)(3y)+(3y)^2=16x^2-24xy+9y^2$.

Example 1.13. $x^2+4x+4=(x+2)^2$.

The techniques introduced so far can also be combined together as showed in the following examples.

Example 1.14. $x^3+6x^2+9x=x(x^2+6x+9)=x(x+3)^2$.

Example 1.15. $(x+y-z)(x+y+z)=[(x+y)-z][(x+y)+z]=(x+y)^2-z^2=x^2+2xy+y^2-z^2$.

Example 1.16. $(x+y+z)^2=[(x+y)+z]^2=(x+y)^2+2(x+y)z+z^2=x^2+2xy+y^2+2xz+2yz+z^2$. *So, the square of a sum of an arbitrary number of terms becomes:*

$$(a+b+c+d+\dots)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd+\dots$$

1.1.4 Cube of a binomial

For cubing the sum and the difference of two quantities a and b, we have the following two rules

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3. \tag{1.4}$$

So, the cube of a sum (difference) of two quantities a and b is equal to the cube of the first term, plus (minus) three times the square of the first term by the second term, plus (plus) three times the first term by the square of the second term, plus (minus) the cube of the second term.

Example 1.17. $(2x+y)^3 = (2x)^3 + 3(2x)^2y + 3\cdot 2x(y)^2 + y^3 = 8x^3 + 12x^2y + 6xy^2 + y^3$.

Example 1.18. $(x^2-3y)^3 = (x^2)^3 - 3(x^2)^2 3y + 3x^2(3y)^2 - (3y)^3 = x^6 - 9x^4y + x^2y^2 - 27y^3$.

Example 1.19. $a^3b^3 - 3a^2b^2 + 3ab - 1 = (ab-1)^3$.

1.1.5 Sum and difference of cubes

Contrary to squares, *both* sum and difference of two cubes can be decomposed. The following rules hold

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2), \quad a^3 - b^3 = (a-b)(a^2 + ab + b^2). \tag{1.5}$$

Note 1.2. The middle of the trinomials is always opposite the sign of the binomial.

Note 1.3. The two trinomials $a^2 - ab + b^2$ and $a^2 + ab + b^2$ are not squares because the double product is not present.

We consider some examples.

Example 1.20. $(x^3-1) = (x-1)(x^2+x+1)$.

Example 1.21. $(8x^3+27y^3) = (2x+3y)(4x^2-6xy+9y^2)$.

1.2 Radicals

In many situations, it is useful to simplify mathematical expressions involving radicals, without trying to rewrite them using decimal approximations. The following example on numerical approximations clarifies the importance of this statement. Suppose we have to compute $(\sqrt{2})^8$. Applying only the exponent properties, we obtain $(\sqrt{2})^8 = \left((\sqrt{2})^2\right)^4 = 2^4 = 16$. On the other hand, if we first approximate $\sqrt{2} \approx 1.4$, and then we raise this approximation to the power of 8 we obtain $(\sqrt{2})^8 \approx \left(1.4\right)^8 = 14.76$. So, the error that we make is not negligible!

The main properties of radicals are listed below

$$
\begin{aligned}
&\sqrt[n]{a^n} = a\,, \quad \left(\sqrt[n]{a}\right)^n = a \quad \text{(definition);}\\
&\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \quad \text{(product property);}\\
&\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad \text{(quotient property);}\\
&\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} \quad \text{(power property);}\\
&\sqrt[n]{a^n b^p} = a\sqrt[n]{b^p} \quad \text{(factor out n-powers);}\\
&\sqrt[np]{a^{mp}} = \sqrt[n]{a^m} \quad \text{(simplify rational exponents).}
\end{aligned}
\tag{1.6}
$$

In previous expressions, it is assumed that a and b are positive real numbers when n and p are even integers[1].

Note 1.4. There is no property linked to the n-th root of a sum of two quantities a and b: $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Note 1.5. It is possible to sum two radicals only if they are similar. It is possible to multiply two radicals only if they have the same index.

Example 1.22. $5\sqrt{8} + 3\sqrt{2} = 5\sqrt{2^2 2} + 3\sqrt{2} = 5 \cdot 2\sqrt{2} + 3\sqrt{2} = 13\sqrt{2}$.

Example 1.23. $3\sqrt{27} - \sqrt{12} + \sqrt{2} = 3\sqrt{3^2 3} - \sqrt{2^2 3} + \sqrt{2} = 3 \cdot 3\sqrt{3} - 2\sqrt{3} + \sqrt{2} = 7\sqrt{3} + \sqrt{2}$.

Example 1.24. $\sqrt{2}\sqrt[3]{2} = \sqrt[6]{2^3}\sqrt[6]{2^2} = \sqrt[6]{8 \cdot 4} = \sqrt[6]{32}$.

1.3 Algebraic fractions

An algebraic function is the *ratio between two polynomials*. For example,

$$\frac{x^3 + xy + y^2 + 2}{x^2 - y}$$

is an algebraic fraction. The methodology used to simplify algebraic fractions is exactly the same used to simplify numerical fractions. Consider the following examples.

Example 1.25. $\dfrac{x^2 + x}{x^2 - 1} + \dfrac{x+2}{x-1} = \dfrac{x\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} + \dfrac{x+2}{x-1} = \dfrac{x+x+2}{x-1} = \dfrac{2x+2}{x-1}$.

Example 1.26. $\dfrac{3x(x+2)}{x+1} \cdot \dfrac{x-1}{x+2} = \dfrac{3x\cancel{(x+2)}}{x+1} \cdot \dfrac{x-1}{\cancel{x+2}} = \dfrac{3x(x-1)}{x+1} = \dfrac{3x^2 - 3x}{x+1}$.

Example 1.27. $\dfrac{x^2 - 1}{x^3 + 1} = \dfrac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(x^2 - x + 1)} = \dfrac{x-1}{x^2 - x + 1}$.

[1] In this course we are mainly interested to the case $n = 2$ (square root) or $n = 3$ (cubic root).

2 Sets, Numbers and Functions

The aim of this chapter is to review certain mathematical concepts and tools which should be known to the reader. In particular, the following concepts are presented: i) the summation symbol, ii) the general concept of set along with various relations and operations, iii) numbers and, in particular, intervals of real numbers, iv) functions. In what follows the symbol $\mathbb{N}$ indicates the set of natural numbers, $\mathbb{Z}$ the set of integer numbers, $\mathbb{Q}$ the set of rational numbers and, finally, $\mathbb{R}$ the set of real numbers.

2.1 The summation symbol

The summation symbol Σ (capital sigma) was introduced around 1820 by the physicist and mathematician J. Fourier (1768-1830). It is a very convenient way to write complicated formulae.
Suppose that we want to write the sum of the integer numbers 1, 2, 3. In this case, we can write $1+2+3$. However, if we want to write the sum of the integer numbers from 1 to 100[1] we (probably) write

$$1+2+\cdots+99+100\,, \tag{2.1}$$

where the dots warn that the summation involves also the numbers from 3 to 98, not explicitly displayed. To represent (2.1) more parsimoniously, we can write

$$\sum_{i=1}^{100} i$$

which reads *the sum of i for i going from 1 to 100*. In general, however, the addends of (2.1) can be more complex. For example, they can be:

- the reciprocal of the natural numbers: $1/i$,
- the square of the natural numbers: i^2,
- any expression involving the natural numbers, such as the ratio between a natural number and its consecutive.
- etc.

Generally speaking, given a finite sequence of terms

$$a_1, a_2, \cdots, a_n \tag{2.2}$$

[1] It is reported that at the age of 7 the Prince of Mathematicians Friedrich Gauss (1777-1855) amazed his teacher by summing the integers from 1 to 100 almost instantly, having quickly spotted that the sum was actually 50 pairs of numbers, with each pair summing to 101, total 5 050.

(which reads *a sub one*, *a sub two*, etc.), to denote their sum we can choose a letter, say i, as an index ranging from m to n, and write

$$\sum_{i=m}^{n} a_i,$$

which reads *the sum of a (sub) i for i (going) from m to n*. a_i is called the *general term*. The value of the sum *does not depend* on the name chosen for the index, but only on its *range*. For this reason, we say that the summation index is a *dummy* index.
In particular the sums

$$\sum_{i=m}^{n} a_i \quad , \quad \sum_{j=m}^{n} a_j \quad \text{e} \quad \sum_{k=m}^{n} a_k$$

are equivalent. The following examples will fix the concepts:

- $\displaystyle\sum_{i=5}^{10} \frac{1}{i^2} = \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2}\,;$
- $\displaystyle\sum_{i=2}^{100} \frac{i}{i-1} = \frac{2}{2-1} + \frac{3}{3-1} + \cdots + \frac{99}{99-1} + \frac{100}{100-1}\,;$
- $\displaystyle\sum_{i=0}^{5} (-1)^i = (-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 = 1 - 1 + 1 - 1 + 1 - 1\ (= 0)\,.$

The summation symbol is subject (intuitively) to the same properties of the sum operation. In particular, the *associative property* holds

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k \qquad (m < n).$$

The following examples conclude this section.

Examples.

- $\displaystyle\sum_{i=2}^{100} \frac{2i+4}{i-1} = 2\sum_{i=2}^{100} \frac{i+2}{i-1}\,;$
- $\displaystyle\sum_{i=0}^{20} \frac{(-1)^i}{i} = (-1)\sum_{i=0}^{20} \frac{(-1)^{i-1}}{i}\,.$

2.2 Sets

A set is identified by explicitly declaring the *objects* (the *elements*) belonging to it, or a *property* which characterises them. Sets are usually denoted by capital letters such as A, B, ..., while their elements are denoted by small letters a, b, ...
A typical symbol in Set theory, which is too important to be given up, is the symbol $\in$, which indicates that an element *belongs* to a set. Writing

$$x \in A \text{ or } A \ni x \tag{2.3}$$

means that the element x *belongs* to the set A. To say, instead, that the element x *does not belong* to the set A the writing

$$x \notin A \text{ or } A \not\ni x \tag{2.4}$$

is used. Two sets are *equal* if and only if they have the same elements. Using the symbol $\forall$ (*for all*), we have the following statement

$$A = B \Leftrightarrow (\forall x\ x \in A \Leftrightarrow x \in B), \tag{2.5}$$

where the symbol $\Leftrightarrow$ stands for *if and only if*. In particular, the ordering with which the elements are listed is not relevant, but only the elements themselves matter.

Among the various sets there is a very special one, without any elements, called the *empty set* and denoted by $\emptyset$. From Equation (2.5) it follows that the empty set is unique.

In general, two representations are used in order to describe a set:

1. *Extensive representation*: all the elements of a set are explicitly listed between curly brackets. For instance, $A = \left\{ 0, \pi, \sqrt{2}, \text{Pordenone}, \text{Forlí} \right\}$.
2. *Intensive representation*: all the elements of a set are implicitly listed through a common property. For instance, $A = \{ x \mid x \text{ is an even natural number} \}$.

Verifying whether an element x belongs to a set A is not a trivial task. Suppose, for example, $A = P$, the set of prime numbers. While it is immediate to verify that $31 \in P$, it is more difficult to check that also $15\,485\,863 \in P$. However, to verify that $2^{43\,112\,609} - 1 \in P$[2] a long computation time is required, even using powerful computers.

We say that A is a *subset* of B, or that A is *contained* (or *included*) in B, or that B is a *superset* of A, if every element in A is also an element of B. We write

$$A \subseteq B \quad , \quad B \supseteq A.$$

The inclusion symbol, $\subseteq$, does not exclude the possibility that A and B coincide. If we want to rule out this possibility, the symbol of *proper* (or *strict*) inclusion must be used, i.e.,

$$A \subset B \qquad B \supset A,$$

which reads *A is strictly included in B*. In particular, the empty set $\emptyset$ is strictly included in every other set. If $A \neq \emptyset$ and $A \subset B$, we also say that A is a *proper* subset of B. Every set A has as *improper* subsets A itself and $\emptyset$. In this course, we are also interested in sets with only one element: if $a \in A$, then $\{ a \} \subseteq A$.

Note 2.1. Symbols $\in$ and $\subset$ have a very different meaning. The first one links two different objects (an element and a set), while the second links objects of the same type (two sets).

Finally, given a set A, the set of all subsets of A is given the name of *power set* and is denoted by $\mathscr{P}(A)$. For instance, if $A = \{ a, b \}$, then

$$\mathscr{P}(A) = \{ \emptyset, \{ a \}, \{ b \}, A \} .$$

[2]This is one of the biggest prime numbers known at the end of 2009, with 12 978 189 digits. We would like to stress that most of the cryptographic algorithms used nowadays are based on the usage of huge prime numbers.

2.3 Relations and operations with sets

Definition 2.1 (Union of two sets)**.** *The* union *of two sets A and B, denoted by $A \cup B$, is the set of elements x such that x belongs to A, to B, or both*[3]

$$A \cup B \stackrel{\text{def}}{=} \{\, x \mid x \in A \vee x \in B \,\}\,.$$

Example 2.1. If $A = \{\, 0, 1, 2, 3 \,\}$ and $B = \{\, 2, 3, 4 \,\}$, then $A \cup B = \{\, 0, 1, 2, 3, 4 \,\}$.

Definition 2.2 (Intersection of two sets)**.** *The* intersection *of two sets A and B, denoted by $A \cap B$, is the set of the elements x such that x belongs to A and to B*

$$A \cap B \stackrel{\text{def}}{=} \{\, x \mid x \in A \wedge x \in B \,\}\,.$$

Example 2.2. Let A and B be as in Example 2.1, then $A \cap B = \{\, 2, 3 \,\}$.

If two sets have an empty intersection, i.e., if $A \cap B = \emptyset$, they are called *disjoint*. The empty set $\emptyset$ is disjoint with itself and with all sets.
The operations introduced above show strong analogies with the arithmetic operations, where the union plays the role of the sum, whereas the intersection plays the role of the product. In particular, the *associative* property, which is immediately verifiable and where A, B and C denote any three sets, holds true

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C).$$

(as a consequence, it is possible to write simply $A \cup B \cup C$ and $A \cap B \cap C$).
Besides, the following relations hold

$$\begin{aligned} A \cup A = A; &\quad A \cap A = A; \\ A \cup B = B \cup A; &\quad A \cap B = B \cap A; \\ A \cup \emptyset = A; &\quad A \cap \emptyset = \emptyset; \\ A \cup B \supseteq A; &\quad A \cap B \subseteq A; \\ A \cup B = A \Leftrightarrow A \supseteq B; &\quad A \cap B = A \Leftrightarrow A \subseteq B. \end{aligned}$$

The *distributive* property reads as

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad , \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Note 2.2. There are, actually, two distributive properties. One of the union with respect to the intersection, and another one of the intersection with respect to the union. Instead, in the arithmetic operations case, only the distributive property of the product with respect to the sum is valid: $a(b + c) = ab + ac$.

Definition 2.3 (Difference between sets)**.** *Given two sets A and B, the* difference *between A and B, denoted by $A \setminus B$ or $A - B$, is the set made up of the elements which belong to A but not to B.*

$$A \setminus B \stackrel{\text{def}}{=} \{\, x \mid x \in A \wedge x \notin B \,\}\,.$$

[3]The symbols $\vee$, *vel*, and $\wedge$, *et*, are commonly used in logic and Set theory. They mean *or* and *simultaneously*, respectively.

Example 2.3. Let A and B as in Example 2.1, then $A \setminus B = \{0, 1\}$.

In particular, if $B \subseteq A$, the set $A \setminus B$ is named *complementary set of B with respect to A*. If the role of A is clear it will be denoted by $\complement_A B$ or $\complement B$. Often, it happens that all of the sets under consideration are subsets of a common set U, called the *universe set*. In this case, we simply write $\complement B$ instead of $\complement_U B$.

The Set theory presented so far is the so called *naive theory*. Although sufficient for our purposes, it presents some problems: in particular, paradoxes can arise[(4)].

We have seen that the sets $\{a, b\}$ and $\{b, a\}$ coincide because they have the same elements. In many practical situations, however, it is important to deal with *ordered* pairs, where the ordering in which the elements are written does matter. More precisely, given two sets A and B, an *ordered pair*, denoted by (a, b), is obtained by choosing an element $a \in A$ and an element $b \in B$ in the specified order. In symbols

$$(a, b) = (a', b') \Leftrightarrow a = a' \wedge b = b' .$$

It is convenient to observe that, in general:

$$\{a, b\} = \{b, a\} \quad \text{while} \quad (a, b) \neq (b, a).$$

Definition 2.4 (Cartesian product)**.** *Given two sets A and B, the* Cartesian product*, or simply* product *of A and B, denoted by $A \times B$, is the set of all the ordered pairs (a, b), with $a \in A$ and $b \in B$. In formulae*

$$A \times B \overset{\text{def}}{=} \{(a, b) \mid (a \in A) \wedge (b \in B)\} .$$

Given the importance of the ordering in the pair (a, b), it should be clear that, whenever A is different from B, $A \times B \neq B \times A$. In the case when $A = B$, you write $A \times A = A^2$.
Finally, you can also consider Cartesian products of more than two sets and, in the case of the Cartesian product of a set with itself n times you write $A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$.

2.4 Numbers

The building blocks of mathematics are numbers. In particular, the following sets of numbers are frequently used

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} .$$

– $\mathbb{N}$ is the set of *natural numbers*. The mathematician Leopold Kronecker (1823-1891) used to say that *natural numbers are God's creation*. In what follows, the set of natural numbers is

$$\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\} .$$

This set has a minimum element, the 0, but not a maximum element. Precisely, every subset of the set of natural numbers admits a minimum element.

[4]Maybe the most famous one is the barber paradox due to Bertrand Russel (1872-1970). The paradox is the following: *The barber is the one who shaves all those, and those only, who do not shave themselves. The question is, does the barber shave himself?*

- $\mathbb{Z}$ (the symbol derives from the German word *zahl*, which means *number*, *digit*) is the set of *integer numbers*. Broadly speaking, integer numbers are natural numbers with sign, with the exception of the 0 ($+0 = -0 = 0$)
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$
Each natural and integer number admits a *consecutive*.
- $\mathbb{Q}$ (the symbol is due to the fact that a rational number is essentially a *quoziente* - Italian for quotient) is the set of *rational number*, i.e., numbers which can be represented as ratios (or *fractions*) of integer numbers, where care is taken not to put the number 0 at the denominator.
$$\mathbb{Q} = \{\, {}^{m}/_{n} \mid m \in \mathbb{Z},\ n \in \mathbb{N},\ n \neq 0 \,\}.$$
There are infinitely many *equivalent* fractions *representing* the same rational number. For instance, $1/7$ is equivalent to $2/14$, to $3/21$, and so on. Among them, it is particularly convenient to consider fractions *reduced to its lowest terms*, where the numerator and the denominator are prime with respect to each other.
Rational numbers admit also a *decimal representation*, illustrated in the following examples. To represent $2/5$ you get $2/5 = 0.4$. But dividing 214 by 495, instead, you obtain $214/495 = 0.4323232\ldots$. In the first case, a *finite* amount of numbers is required after the decimal point, whereas, in the second case, the digits 32 repeat indefinitely. It is said that the alignment is *periodical* and that 32 is the *period*. Differently from the previous two sets of numbers, you *cannot* speak about the *consecutive* of a rational number. In particular, between two rational numbers there is an infinite number of rational numbers:
$$\text{if } a = \frac{m}{n} \text{ and } b = \frac{p}{q} \text{ than the number } c = \frac{a+b}{2} \text{ is a rational number between } a \text{ and } b.$$
- $\mathbb{R}$ is the set of *real numbers*. In this course, we do not want to describe rigorously this set of numbers. Broadly speaking, the set of real numbers can be thought as the set of all integer numbers, fractions, radicals, the numbers as π, etc.

The following relations hold
$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$
Common to all of these sets, there is the possibility to do sums and products. However, it is not always possible to perform subtractions in $\mathbb{N}$ and divisions in $\mathbb{Z}$. Sometimes, it may happen that we have to use the set of complex numbers. This set is denoted by $\mathbb{C}$ and it is a superset of $\mathbb{R}$. The main advantage is that, within the set of complex numbers, it is always possible to compute the square root of a negative number.

2.5 Intervals of real numbers

Some subsets of $\mathbb{R}$, which are called *intervals*, deserve special attention: In this section, we give the definition and we consider some properties of these subsets.

Definition 2.5. *Given two real numbers a and b, with $a < b$, you call* intervals *the following subsets of* $\mathbb{R}$

$]a,a[=(a,a)$	$\emptyset$	*empty interval*
$]a,b[=(a,b)$	$\{\,x \mid a < x < b\,\}$	*bounded interval* open
$[a,b]$	$\{\,x \mid a \le x \le b\,\}$	*bounded interval* closed
$[a,b[=[a,b)$	$\{\,x \mid a \le x < b\,\}$	*bounded interval*left-closed *and* right-open
$]a,b]=(a,b]$	$\{\,x \mid a < x \le b\,\}$	*intervallo limitato* left-open *and* right-closed
$]a,+\infty[=(a,+\infty)$	$\{\,x \mid x > a\,\}$	*left-bounded and right-unbounded* left-open
$[a,+\infty[=[a,+\infty)$	$\{\,x \mid x \ge a\,\}$	*left-bounded and right-unbounded* left-closed
$]-\infty,a[=(-\infty,a)$	$\{\,x \mid x < a\,\}$	*left-unbounded and right-bounded* right-open
$]-\infty,a]=(-\infty,a]$	$\{\,x \mid x \le a\,\}$	*left-unbounded and right bounded* right-closed

The numbers a and b are the extremes *of the interval. Bounded intervals are also named* segments, *whereas unbounded intervals* half lines.

Consistently, we can write $\mathbb{R} =]-\infty,+\infty[$ or, equivalently, $\mathbb{R} = (-\infty,+\infty)$, which is the only unbounded interval whose geometrical image is the entire straight line. It is both open and closed. When $a = b$ the closed interval $[a,a]$ has only one element and it is named *degenerate* interval. Sometimes, also the empty set $\emptyset$ is considered as an interval. In particular, it is named *empty interval*.

If $[a,b]$ is a bounded interval, i.e. $[a,b] \stackrel{\text{def}}{=} \{x \in \mathbb{R} : a \le x \le b\}$ the point

$$x_0 = \frac{a+b}{2}$$

is named *center* while the number

$$\delta = b - x_0 = x_0 - a$$

radius. In particular, an open interval with center x_0 and radius δ is given by

$$]x_0 - \delta\,,\, x_0 + \delta[\,.$$

Each point of an interval that does not coincide with the (*possible*) extremes is named *interior point*.

2.6 Functions

In applications, relations between two sets A and B have great interest. Among these relations, *functions* between two intervals of real numbers deserve particular attention. The following definition holds.

Definition 2.6. *Given two sets A and B, a* function *from A to B is a law that associates with* each *element of A one* (single) *element of B. The set A is called the* domain *of the function, while the set B is called the* co-domain. *If x is an element of the set A and y is the* unique *element of the set B corresponding to A, it is said that y is* function *of x; in symbols $y = f(x)$.*

It is important to remind that, in order to assign a function it is necessary to specify

- the domain

– the co-domain
– a law that associates with the element x in the domain the unique element y in the co-domain.

Different notations are used to indicate a function. The most complete one is the following

$$f : A \to B,\ x \mapsto f(x),$$

but, often, the symbol

$$x \mapsto f(x),$$

is used if the sets A and B have already been defined or their definition is clear from the context. However, the most used one is the less rigorous notation $y = f(x)$ (to be read *y is f of x*).

Example 2.4. If A and B are the set of real numbers, we can consider the function that associates with the real number $x \in A$ the element $y = x^2$ in B. So, we can use one of the following three notations

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2,$$

or

$$x \mapsto x^2$$

or

$$y = x^2.$$

Often, to visualize a function arrow diagrams are used. For instance

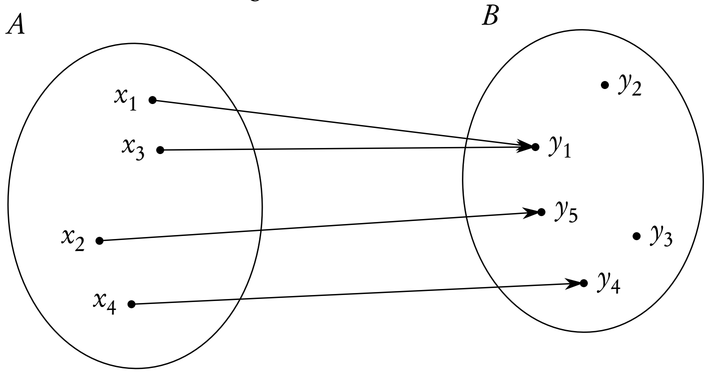


Figure 2.1 *Arrow diagram to visualize a function between two finite sets*

In particular, it *is required* that from *each* point of A originates *exactly* one arrow. Conversely, it can happen that a point of B is "shot" with more than one arrow, or that it is not "shot" at all.
In applications, the so-called *image* set has a particular interest. It is given by

$$I \subseteq B \overset{\text{def}}{=} \{\, y \in B \mid \exists x \in A,\ y = f(x) \,\}, \tag{2.6}$$

and defined as the set of all possible *outputs* $y \in B$ coming from all possible *inputs* $x \in A$. The set I is denoted by the symbol $f(A)$. If C is a subset of A, then we can consider the set $f(C) \subseteq f(A)$.

Other types of representations can be used for functions. For instance, let f be the function that associates with the element x in $A = \{1, 2, 3, 4, 5\}$ the element y in $B = \{1/2, 1, 3/2, 2, 5/2\}$ (note that the domain of f is the subset of natural numbers $\{1,2,3,4,5\}$, whereas the co-domain is the set of rational numbers). The following tabular representation can be used

x	$x/2$
1	$1/2$
2	1
3	$3/2$
4	2
5	$5/2$

.

Table 2.1 *Tabular representation of a function*

In particular, in the first column there are the natural numbers 1, 2, ..., 5 whereas in the second the corresponding halves.

Another type of representation is the *pie chart*. For instance, let us consider an undergraduate program where 120 students, coming from different provinces, are enrolled:

Gorizia	Pordenone	Treviso	Trieste	Udine
5	70	15	10	20.

To build the pie-chart, first we compute the percentages relative to each province

Gorizia	Pordenone	Treviso	Trieste	Udine
4.17	58.33	12.5	8.33	16.67

,

then we map these percentages into the slices of the pie chart, taking into account that the whole pie measures 360°:

Gorizia	Pordenone	Treviso	Trieste	Udine
15°	210°	45°	30°	60°

At this point the chart is immediate

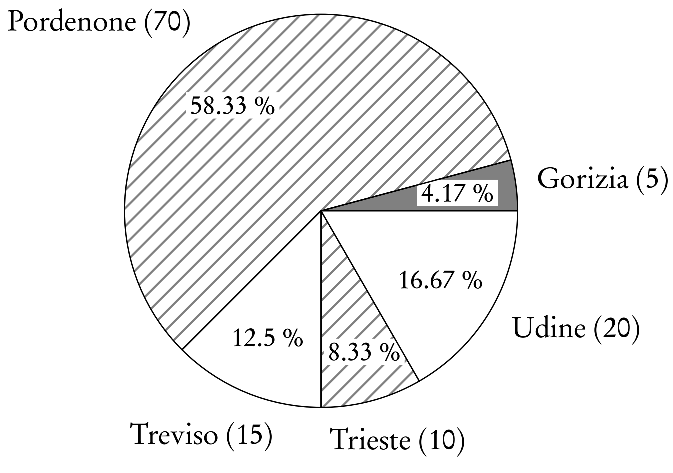


Figure 2.2 *Pie-chart indicating the origin of* 120 *students enrolled in an undergraduate course*

Finally, a bar-chart it is also used

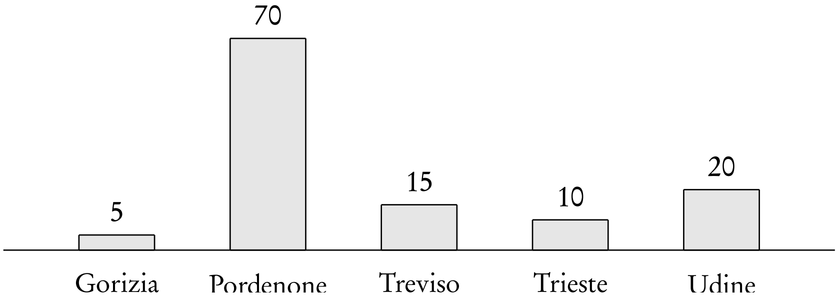


Figure 2.3 *Bar-chart indicating the origin of* 120 *students enrolled in an undergraduate course*

As mentioned, we are mainly interested in numerical functions, where *input* and *output variables* are numbers or groups of numbers. In particular, in this course real numbers come into play as variables, and for this reason we will talk about *real functions of a real variable*. In all these cases, we are dealing with laws associating with a real number x one real number y only, so that they have a subset A of $\mathbb{R}$ as domain and $\mathbb{R}$ as co-domain.
In order to visualize the behaviour of a function, the study of its *graph* in an appropriate Cartesian plane turns out to be very useful. In particular, the following definition holds

Definition 2.7. *The graph of a function $f : A \to B$ is the set of pairs (x, y), with $x \in A$, $y \in B$, such that $y = f(x)$.*

For instance, if we consider example in Table 2.1, we have to represent the points

$$A = (1, 1/2),\ B = (2, 1),\ C = (3, 3/2),\ D = (4, 2),\ E = (5, 5/2),$$

into the following *graph*:

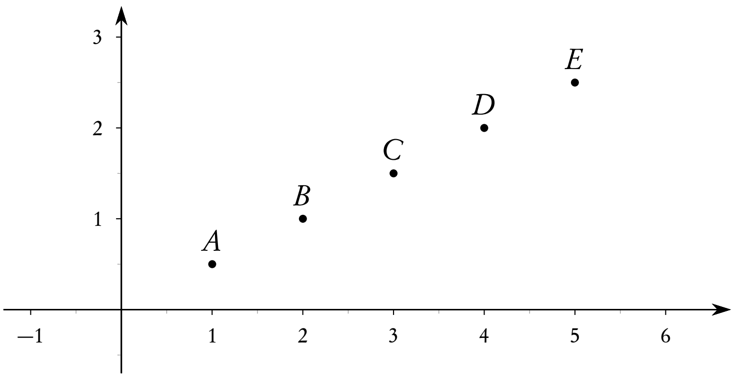


Figure 2.4 *Cartesian graph*

The graph in Figure 2.4 can be thought as a compacted arrow diagram.

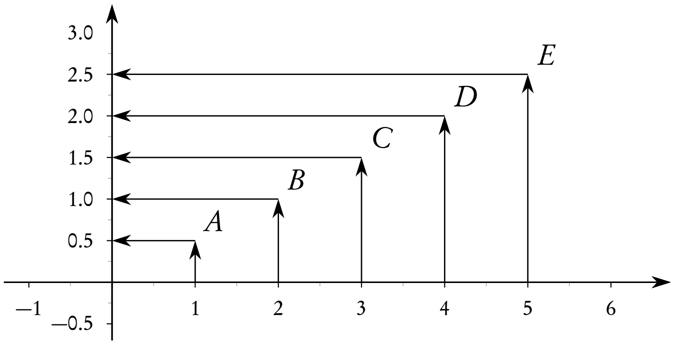


Figure 2.5 *Cartesian graph, with arrows.*

Representing a function into a Cartesian graph has many advantages. This is clear if we compare Figure 2.4 with Table 2.1. In particular, from the graph it is immediate to figure out that the function is *monotonic* (precisely, it is a monotonically increasing function) and that its growth is *steady*. The advantages are more evident if we consider the function that associates with the real number $x \in \mathbb{R}$ the element $y = x/2$ in $\mathbb{R}$. In this case, the variable x assumes an infinite number of values and so it is not possible to have a tabular representation[5]. In particular, we have the following Cartesian graph:

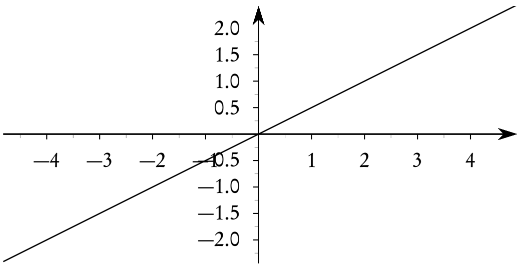


Figure 2.6 *Cartesian graph of the function* $y = {}^{x}/_{2}$

Obviously, graph in Figure 2.7 contains also the points represented in the graph of Figure 2.4:

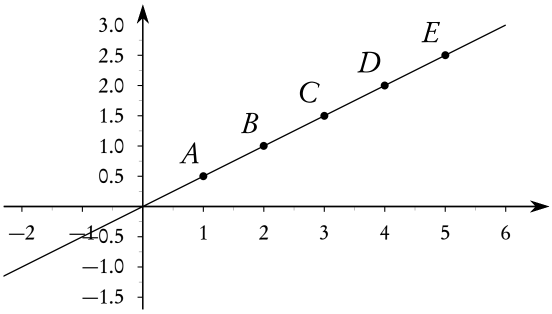


Figure 2.7 *Cartesian graph of the function* $y = {}^{x}/_{2}$. *Some points are put in evidence.*

[5]Note that, however, the law that associates with the element x the element y is the same of the previous case: In order to assign a function, *it is not* sufficient to define only the law but also the domain and the co-domain.

Often it is required to study a real function of a real variable in order to draw an indicative graph of it. Some software are devoted to this end[6]. However, we have to keep in mind that computers are not build up to solve all our problems! For instance, let consider the graph of the following function

$$f(x) = \sin {}^1\!/\!_x .$$

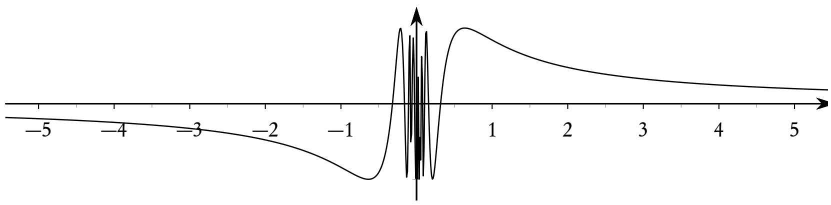


Figure 2.8 *Graph of* $f(x) = \sin {}^1\!/\!_x$

It should be clear that as x approaches 0, the graph in Figure 2.8 is not very significant. So, we try to zoom the horizontal axis:

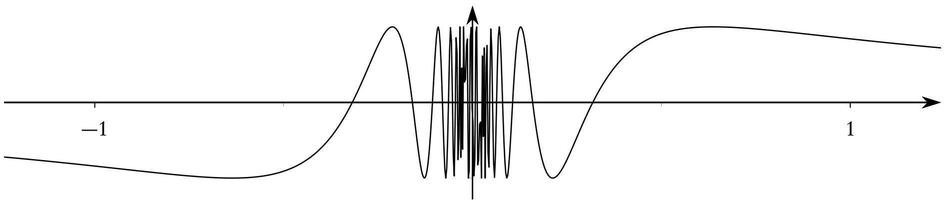


Figure 2.9 *Graph of* $f(x) = \sin {}^1\!/\!_x$. *We zoom the horizontal axis.*

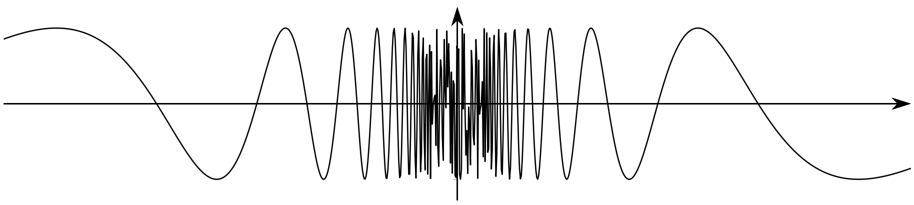


Figure 2.10 *Graph of* $f(x) = \sin {}^1\!/\!_x$. *We zoom the horizontal axis of the graph in Figure 2.9.*

without success. Le us now consider a luckier example. Precisely, the graph of the function $f(x) = x^3 - 3x^2$ is obtained, with sufficient accuracy, with a commercial software and displayed in the following figure.

[6]Among the commercial software, we point out two sophisticated, although complicated, packages: *Mathematica* and *Maple*. Instead, *Maxima* (a less sophisticated version of *Mathematica*) and *Geogebra* are two non commercial software. In particular, most of the graphs in these notes are obtained with *Geogebra*.

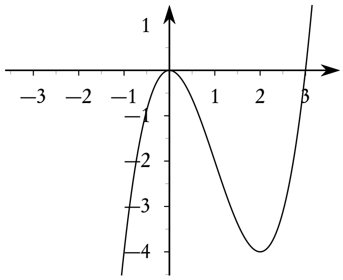


Figure 2.11 *Grafico di* $f(x) = x^3 - 3x^2$

Observing the graph, it is evident that the function $f(x) = x^3 - 3x^2$ is: i) strictly increasing on the interval $]-\infty, 0]$, ii) strictly decreasing on the interval $[0, 2]$ and, iii) strictly increasing on the interval $[2, +\infty[$.

All the graphs, except those in Figure 2.9 and 2.10, present the same unit of measure on both axes. Cartesian systems of this type are named *mono-metric*. However, in practice, it is not always possible to use mono-metric Cartesian systems. For instance, both graphs in Figure below represent a circumference center at the origin and with unitary radius. However, only the left panel employs a mono-metric Cartesian system.

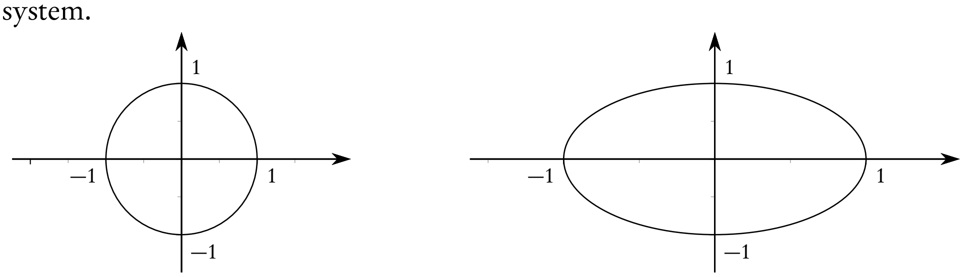


Figure 2.12 *Circumference with center at the origin and unitary radius. The left panel employs a mono-metric Cartesian system. The right one, instead, has not a Cartesian system and it appears warped.*

2.7 Exercises

Exercice 2.1. *Given the sets* $A =]-\infty, 2]$, $B = \{1, 2\}$ *e* $C = [0, 5[$, *determine*

1. $(A \setminus C) \cup B$;
2. $(A \setminus B) \cup C$;
3. $(A \setminus B) \cap C$;
4. $(C \setminus B) \cap A$.

Exercice 2.2. *Given the sets* $A = \{1\}$, $B =]-1, 2[$ *e* $C =]0, +\infty[$, *determine*

1. $(A \cup C) \cap B$;
2. $A \setminus C$;
3. $(C \setminus A) \cap B$;
4. $(C \cup B) \setminus A$;
5. $(b \setminus A) \cap C$.

Exercice 2.3. *Discuss succinctly, yet unequivocally, the following questions.*

1. *Is it possible to find three sets* A, B, C *such that* $(A \cap B) \cup C = \emptyset$*?*
2. *Is it possible to find three sets* A *e* B *such that* $A \cap B = A$*?*
3. *Is it possible to find three sets* A, B, C *such that* $(A \cap B) \cup C = A$*?*
4. *If* $A \subseteq B$ *then* $(C \setminus B) \subseteq (C \setminus A)$.

3 Equations

3.1 Linear equations

In this section we look at linear equations in one variable x.
The most general *linear* equation – this means there will be no x^2 terms and no x^3's, just x's and numbers – in *one variable* is of the type

$$ax = b \quad , \quad a \neq 0 . \tag{3.1}$$

Equation in (3.1) admits a unique solution[(1)]

$$x = \frac{b}{a} . \tag{3.2}$$

If $a \in \mathbb{R}$, then three cases must be considered:

- $a \neq 0$: the equation has only one solution $x = b/a$;
- $a = 0 \wedge b \neq 0$: the equation has no solution ;
- $a = 0 \wedge b = 0$: the equation has an infinite number of solutions (basically all $\mathbb{R}$).

In particular, it is important to consider the above conditions when solving parametric equations. For instance, consider the following example.

Example 3.1. Solve the following equation:

$$(a^2 - 1)x = a + 1 .$$

To solve it, we have to take into account the following cases:

- If $a \neq \pm 1$, than it has only one solution $x = (a+1)/(a^2-1) = 1/(a-1)$;
- If $a = 1$, than it has no solution;
- If $a = -1$, than it has an infinite number of real solutions.

The most general *linear* equation in *two variables* is of the type

$$ax + by = c \quad , \quad (a, b) \neq (0, 0) . \tag{3.3}$$

The condition on the parameters a and b is equivalent to say that they are not both zero at the same time. Equation (3.3) has always an infinite number of solutions. To obtain these solutions one first

[1] The *Fundamental Theorem of Algebra* states that an equation of grade n has at maximum n solutions in $\mathbb{R}$. As a consequence equation (3.1) has always one solution. This is not the case if we consider non linear equations. For this type of equations it is possible to have a number of solutions lower than the grade.

transforms equation (3.3) into a linear equation in one variable by fixing one of the two variables, then solves the latter as previously discussed. For instance, the following equation

$$2x+3y=1$$

has as solutions the pairs $(0, 1/3)$, $(1/2, 0)$, $(-1, 1)$, etc.

The following one variable equation, can be considered as a two variables equation where the coefficient of y is 0:

$$3x=1,\text{ or } 3x+0y=1\,,$$

and it has as solutions the pairs $(1/3, 1)$, $(1/3, 2)$, $(1/3, -5)$, etc.

3.2 Two dimensional systems of linear equations

In mathematics, a *system of linear equations in two variables* is a collection of two linear equations involving the same set of variables. For example

$$\left\{\begin{array}{rcl} ax+by &=& p \\ cx+dy &=& q, \end{array}\right. \tag{3.4}$$

is a system of linear equations in the two variables x and y. The *grade of the system* is obtained as product of the grades of each equation. In particular, the system in Equation (3.4) has grade equal to one.

The word *system* indicates that equations have to be considered collectively, rather than individually. A *solution* to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied. In particular, the system is said

- *determinate* if it has a unique solution;
- *indeterminate* if it has an infinite number of solutions;
- *inconsistent* if it has no solution.

In general, a system of linear equations is *consistent* if there is at least one set of values for the unknowns that satisfies every equation in the system.

Consider the following examples:

- $\left\{\begin{array}{l} 2x+y=1 \\ x-y=2 \end{array}\right.$: The system is *consistent* and *determinate*. It has as unique solution the pair $(1,-1)$.
- $\left\{\begin{array}{l} x-2y=1 \\ 2x-4y=2 \end{array}\right.$: The system is *consistent* and *indeterminate*. It has as solutions the pairs $(2t+1, t)\, \forall t \in \mathbb{R}$.
- $\left\{\begin{array}{l} x-2y=1 \\ 2x-4y=3 \end{array}\right.$: The system is *inconsistent*.

One method of solving a system of linear equations in two variables is the *by substitution* method. The method of solving *by substitution* works by solving one of the two equations (we choose which one) for one of the variables (we choose which one), and then plugging this back into the other equation, *substituting* for the chosen variable and solving for the other. Then we back-solve for the first variable. For instance

$$\begin{cases} 2x+y=1 \\ x-y=2 \end{cases}, \quad \begin{cases} y=1-2x \\ x-y=2 \end{cases}, \quad \begin{cases} y=1-2x \\ x-(1-2x)=2 \end{cases}, \quad \begin{cases} y=1-2x \\ x=1 \end{cases}, \quad \begin{cases} y=-1 \\ x=1 \end{cases}.$$

3.3 Second order equations

This section is about *single-variable quadratic equations* and their solutions.
The most general *quadratic equation* is any equation having the form

$$ax^2+bx+c=0, \quad a\neq 0, \tag{3.5}$$

where x represents an unknown, and a, b, and c represent known numbers such that a is not equal to 0. If $a=0$, then the equation is linear, not quadratic. A quadratic equation can be solved using the general *quadratic formula*

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{3.6}$$

In particular, it admits

- two distinct solutions if the quantity $\Delta=b^2-4ac$ (named *discriminant* or simply *Delta*) is greater than zero;
- one solution (it is said that the quadratic equation has *two coincident real solutions* or that it has a *double solution*) if $\Delta=0$;
- no solution in $\mathbb{R}$ if $\Delta<0$. In this case there are two solutions in the complex set $\mathbb{C}$.

To fix ideas, we consider the following examples.

Examples.

- $2x^2-3x-5=0 \implies x_{1,2}=\dfrac{3\pm\sqrt{9-4\cdot 2(-5)}}{2\cdot 2}=\dfrac{3\pm\sqrt{49}}{4}=\left\langle \begin{matrix} 5/2 \\ -1 \end{matrix}\right.$;
- $x^2-6x+9=0 \implies x_{1,2}=\dfrac{6\pm\sqrt{36-4\cdot 9}}{2}=3$;
- $x^2-2x+2=0 \implies$ there is no solution because $\Delta=4-4\cdot 2<0$.

3.4 Higher order equations

There are general formulae for solving cubic (third degree polynomials) and quartic (fourth degree polynomials) equations. However, we are not interested to them in this course. Instead, there are no formulae to solve general equations having grade greater than 4. We limit our analysis to two simple cases.

3.4.1 Elementary equations

An elementary equation is an equation of type

$$ax^n + b = 0, \quad a \neq 0. \tag{3.7}$$

In order to solve Equation (3.7), we have to make the following steps: i) "take" the term b to the other side of the equation (while changing the sign), ii) divide the latter term by a, iii) find the n-th root of the term $-b/a$. It is important to make attention if n is an odd or an even number.

Example 3.2. $2x^3 + 54 = 0 \implies x^3 = -27 \implies x = -3.$

Example 3.3. $3x^3 - 12 = 0 \implies x^3 = 4 \implies x = \sqrt[3]{4}.$

Example 3.4. $2x^4 + 15 = 0 \implies x^4 = -15/2 \implies$ There is no solution.

Example 3.5. $3x^4 - 14 = 0 \implies x^4 = 14/3 \implies x = \pm\sqrt[4]{14/3}.$

3.4.2 Factorizable equations

We only consider some example.

Example 3.6. $x^3 - x^2 = 0 \implies x^2(x-1) = 0 \implies x^2 = 0 \vee x - 1 = 0 \implies x = 0 \vee x = 1.$

Example 3.7. $x^3 - 1 = 0 \implies (x-1)(x^2+x+1) = 0 \implies x = 1$ only, as the equation $x^2+x+1 = 0$ has no solution ($\Delta < 0$).

Example 3.8. $x^4 - 1 = 0 \implies (x^2-1)(x^2+1) = 0 \implies (x-1)(x+1)(x^2+1) = 0 \implies x = \pm 1.$ (The equation $x^2 + 1 = 0$ has no solution).

3.5 Equations with radicals

A "radical" equation is an equation in which at least one variable expression is stuck inside a radical (in this course we consider the case of square roots or cubic roots).
There is no standard technique to solve radical equations. In general, we solve equations by isolating the variable. So, in the radical equation case we first have to isolate the square (or cubic) root, then to square (or to cube) both members. The new equation does not contain any radical. It is important to remind that, in the square root case, we have to check if all the solutions of the new equation are consistent with the initial one. There are no checks to do in the cubic root case. For instance

Example 3.9. $\sqrt{x+2} + x = 0, \quad \sqrt{x+2} = -x, \quad x + 2 = x^2, \quad x^2 - x - 2 = 0,$

$x_{1,2} = \dfrac{1 \pm \sqrt{1+8}}{2} = \Big\langle \begin{matrix} -1 \\ 2 \end{matrix}$, only the solution $x = 1$ is consistent.

Example 3.10. $\sqrt{x+2} - x = 0, \quad \sqrt{x+2} = x, \quad x + 2 = x^2, \quad x^2 - x - 2 = 0,$

$x_{1,2} = \dfrac{1 \pm \sqrt{1+8}}{2} = \Big\langle \begin{matrix} -1 \\ 2 \end{matrix}$, only the solution $x = 2$ is consistent.

Example 3.11. $\sqrt{1+x^2} = x + 2, \quad 1 + x^2 = (x+2)^2, \quad 4x + 3 = 0, \quad x = -3/4$, the solution is consistent.

Example 3.12. $\sqrt{2x^2+1} = 1 - x, \quad 2x^2 + 1 = (x+2)^2, \quad 2x^2 + 1 = 1 - 2x + x^2, \quad x^2 + 2x = 0,$
$x_1 = -2, x_2 = 0$, both solutions are consistent.

Example 3.13. $\sqrt[3]{x^2-x-1}=x-1$, $x^2-x-1=x^3-3x^2+3x-1$, $x^3-4x^2+4x=0$, $x(x^2-4x+4)=0$,
$x=0 \vee x=2$, both solutions are consistent.

4 Basic notions of Geometry

The aim of this chapter is to review some fundamental concepts of analytic geometry, also known as coordinate geometry, or Cartesian geometry.

4.1 Cartesian coordinates

We start by considering the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$, i.e. the set of all ordered triples of real numbers. Thanks to the one-to-one correspondence between real numbers and points on a straight line, it is possible to represent the elements of $\mathbb{R}^3$ as points on a space, which takes the name of *Cartesian space*. In order to do so, we fix three oriented straights lines, which are called the *Cartesian axes* and usually are taken to be perpendicular to each other. In latter case, we speak about *orthogonal Cartesian space*. Besides, if all the axes have the same unity of measure the Cartesian space is named *mono-metric*. In what follows, we always take into account *mono-metric and orthogonal Cartesian spaces*. The point $(0,0,0)$ corresponds to the intersection point between the axes (called the *origin*). In this way, Cartesian axes are indicated by O_x, O_y, O_z, or, simply x, y, z, and xy, xz, yz are named *Cartesian planes*.

Instead, if we consider the *Cartesian plane* $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, the Cartesian axis O_x is called *horizontal axis* or *axis of abscissae*, whereas O_y *vertical axis* or *axis of ordinates*. The Cartesian plane is denoted by Oxy and the Cartesian space by $Oxyz$. Once the reference system $Oxyz$ is fixed, a one-to-one correspondence is set up between a point P in the space and an ordered triple of real numbers (the *coordinates* of the point), as shown in Figure 4.1. In order to indicate the coordinates of the point P we write $P(x,y,z)$ ($P(x,y)$ into the plane) or, sometimes, $P = (x,y,z)$ ($P = (x,y)$ into the plane).

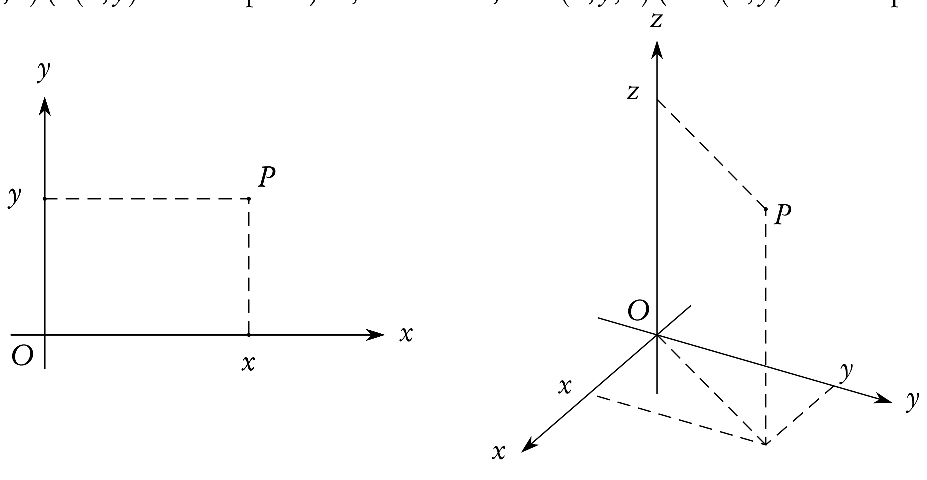


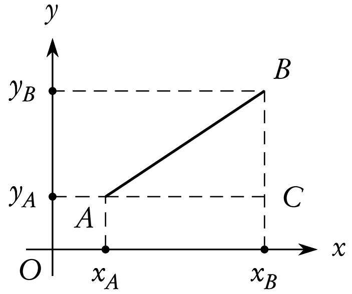
Figure 4.1 *Cartesian coordinates of a point into the plane and into the space.*

4.2 Fundamental formulae of Geometry

Let A and B two points into the Cartesian space Oxy, with respective coordinates (x_A, y_A) and (x_B, y_B). The *distance* between A and B is given by

$$\overline{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}. \tag{4.1}$$

This formula is a direct consequence of the Pythagorean theorem, so the fact that we are working with an orthogonal Cartesian plane is crucial, as shown in Figure 4.2 below.



$$\overline{AC} = x_C - x_A = x_B - x_A$$
$$\overline{BC} = y_B - y_C = y_B - y_A$$

Figure 4.2 *Distance between two points and Pythagorean theorem*

The *midpoint* M of the line segment joining the points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ has Cartesian coordinates (x_M, y_M) given by

$$x_M = \frac{x_A + x_B}{2}, \quad y_M = \frac{y_A + y_B}{2}. \tag{4.2}$$

Finally, we report the formula for the centroid of a triangle. Precisely, given three vertices $A = (x_A, y_A)$, $B = (x_B, y_B)$ and $C = (x_C, y_C)$ of a triangle ABC, the coordinates of the centroid G are given by

$$x_G = \frac{x_A + x_B + x_C}{3}, \quad y_G = \frac{y_A + y_B + y_C}{3}. \tag{4.3}$$

4.3 Lines

The most general equation for a straight line into a Cartesian plane is given by

$$ax + by + c = 0 \tag{4.4}$$

where a and b represent known numbers. In particular a and b cannot be equal to zero at the same time. In order to draw a line into a Cartesian plane it is sufficient to find out two solutions of Equation (4.4).

Example 4.1. Draw into the Cartesian plane the following line: $3x + 2y - 6 = 0$. It is immediate to verify that the two points $(0, 3)$ and $(2, 0)$ satisfy the equation. The graph is the following:

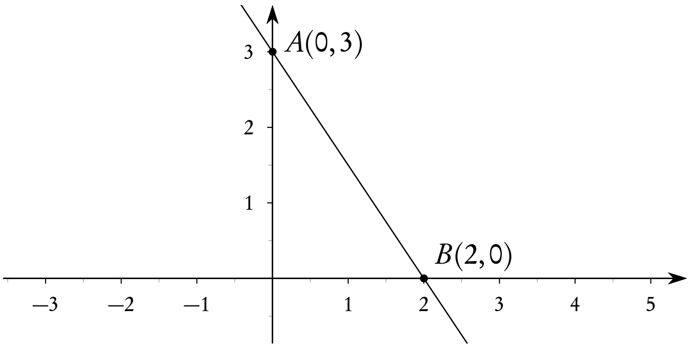


Figure 4.3 *Line* $3x+2y-6=0$

If $b \neq 0$, Equation (4.4) can be rewritten as

$$y = -\frac{a}{b}x - \frac{c}{b},$$

or, setting $m = -\frac{a}{b}$ and $q = -\frac{c}{b}$, as

$$y = mx + q. \tag{4.5}$$

The number m is the line's *slope* (or *gradient*), whereas the number q is called *vertical intercept*. For example, the line in Figure (4.3) can be written as

$$y = -\frac{3}{2}x + 3,$$

with $m = -3/2$ and $q = 3$. We observe that

$$m = \frac{y_B - y_A}{x_B - x_A}. \tag{4.6}$$

The property in previous Equation (4.6) holds for any pair of points of the line. In particular, the numerator $y_B - y_A$ and the denominator $x_B - x_A$ indicate, respectively, the vertical and the horizontal movement done when moving from the point A to the point B. The line is: i) increasing, i.e. the slope m is positive, if it goes up from left to right, ii) decreasing, i.e. the slope m is negative, if it goes down from left to right, iii) horizontal, i.e. the slope m is zero, if it is a constant function. The usual formula for the slope is the following one

$$m = \frac{\Delta y}{\Delta x}, \tag{4.7}$$

where *the difference* $y_B - y_A$ is indicated by Δy (it reads as *delta y*) and the difference $x_B - x_A$ is indicated by Δx (it reads *delta x*). This is a very important notation and it is commonly used in practice. In particular, if we consider any variable g, then the difference between two values of g is named *variation* and it is denoted by Δg. If Δg is positive (negative) we speak about *increment* (reduction) of g. For example, if the company Alfa has a gain g of 150.000 \$ for 2008 and of 180.000 \$ for 2009, then $\Delta g = 30.000$ \$. In other words, the company has performed a gain of 30.000 \$ over one year.

Vertical lines are characterized by equations of type $x = k$, i.e. $b = 0$, while horizontal lines by equations the type $y = k$, i.e. $m = 0$. Besides, two non vertical lines are parallel if and only if they have the same slope, whereas they are perpendicular if and only if the slope of the second line is the negative reciprocal of the slope of the first line, i.e. the product between their slopes is -1. In order to find out the equation of a line, we can encounter the following situations.

1. *Line passing through a point P and known slope*: If $P(x_P, y_P)$ is the point and m is the known slope, than the equation of the line is

$$y - y_P = m(x - x_P). \tag{4.8}$$

2. *Line passing through two points A and B*: if $A(x_A, y_A)$ and $B(x_B, y_B)$ are two points, then the equation of the line is

$$(x - x_A)(y_B - y_A) = (y - y_A)(x_B - x_A). \tag{4.9}$$

Example 4.2. Determine the equation of the line s passing through $(1,2)$ and parallel to the line $r: 2x - y + 5 = 0$.

We first have to determine the slope of the line r. To do this, we write it into the form $y = 2x - 5$: its slope is equal to 2. So, the equation of the line s is $y - 2 = 2(x - 1)$ (or, equivalently $2x - y = 0$).

Example 4.3. Determine the equation of the line s passing through $(2,1)$ and perpendicular to the line $r: x - 2y - 1 = 0$.

We have: $r: y = 1/2\, x - 1/2$. The slope of the line r is $\frac{1}{2}$. Hence, the slope of the line s will be -2 and so the equation of the line is $y - 1 = -2(x - 2)$ (or, equivalently $2x + y - 5 = 0$).

Example 4.4. Determine the line passing through $(2,3)$ and $(4,-1)$.

We immediately obtain $(x - 2)(-1 - 3) = (y - 3)(4 - 2)$, which simplifies as $2x + y - 7 = 0$.

4.4 Parabolas

4.4.1 Parabola with vertical axis

The equation of a parabola with vertical axis is given by

$$y = ax^2 + bx + c, \quad a \neq 0, \tag{4.10}$$

where a, b and c are known numbers, with $a \neq 0$. It has the following fundamental characteristics:

- If $a > 0$ its concavity is upward; if $a < 0$ its concavity is downward.
- The vertex V has abscissa

$$x_V = -\frac{b}{2a}. \tag{4.11}$$

- The ordinate of V can be found by substituting the abscissa x_V into the equation of the parabola.

4.4.2 Parabola with horizontal axis

The equation of a parabola with horizontal axis is given by

$$x = ay^2 + by + c, \quad a \neq 0. \tag{4.12}$$

where a, b and c are known numbers, with $a \neq 0$. It has the following fundamental characteristics:

- If $a > 0$ its concavity is rightward; if $a < 0$ its concavity is leftward.
- The vertex V has ordinate

$$y_V = -\frac{b}{2a}. \tag{4.13}$$

- The abscissa of V can be found by substituting the ordinate y_V into the equation of the parabola.

We present now the following examples.

Example 4.5. $y = 2x^2 - x - 1$. It has an upward concavity; the vertex has coordinates $(1/4, -9/8)$. It intersects the y axis in $(0, -1)$ and the x-axis in $(-1/2, 0)$ (these points are obtained by using the *quadratic formula* for *quadratic equations*). The graph of the parabola $y = 2x^2 - x - 1$ is the following:

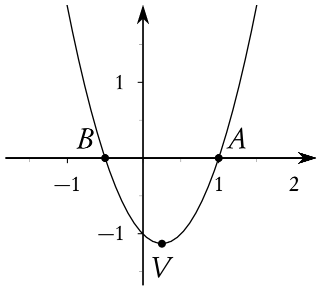


Figure 4.4 *Graph of the parabola of equation* $y = 2x^2 - x - 1$

Example 4.6. $x = y^2 - 2y + 2$. It has a rightward concavity; the vertex has coordinates $(1, 1)$. It intersects the x axis in $(2, 0)$. To find out the intersections with the y axis we have to put x equal to 0. However, equation $y^2 - 2y + 2 = 0$ has no solution (the Δ is negative). We determine other points, $(2, 2)$ and $(5, -1)$, in order to draw the graph of the parabola:

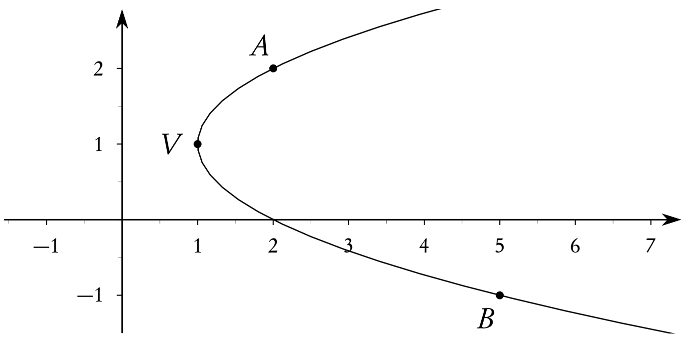


Figure 4.5 *Graph of the parabola of equation* $x = y^2 - 2y + 2$

5 Inequalities

A one-variable inequality is an expression of the type

$$f(x) \lesseqgtr g(x) \quad (\text{or, equivalently } f(x) < g(x) \vee f(x) \le g(x) \vee f(x) > g(x) \vee f(x) \ge g(x)),$$

while a two-variables inequality reads as

$$f(x,y) \lesseqgtr g(x,y) \quad (\text{or, equivalently } f(x,y) < g(x,y) \vee f(x,y) \le g(x,y) \vee f(x,y) < g(x,y) \vee f(x,y) \ge g(x,y)).$$

Solving an inequality means to find a range, or ranges, of values that the variable(s) can take and still satisfy the inequality.
For instance, in the following examples

Example 5.1.
- $3x^2 - 2x > 1$: 2 is a solution; 0 is not a solution.
- $x^2 - 2y^2 \ge x + y$: the pairs $(2,0)$ is a solution; the pair $(2,1)$ is not a solution.

It is important to note that, *usually*, in the one-variable inequalities case, there is an infinite number of solutions which can be represented by using subsets of the real numbers (see Chapter 2, Paragraph 2.5). Often, it is convenient to represent solutions graphically. This is because it is not always possible to express the set of solutions in an analytical way.

5.1 First order inequalities

5.1.1 First order one-variable inequalities

A first order one-variable inequality assumes one of the following forms

$$ax + b > 0, \quad ax + b \ge 0, \quad ax + b < 0, \quad ax + b \le 0. \tag{5.1}$$

In general, it is convenient to consider the case $a > 0$, possibly changing the sign of both members. *Attention: multiplying or dividing both sides of an inequality by the same* negative *number the sign of the inequality changes*. To solve the inequality, then, one has first to isolate the variable, then to divide b by a. The following examples help to fix ideas.

Example 5.2. $3x + 2 \le 0$: $3x \le -2$, $x \le -2/3$, or $x \in]-\infty, -2/3]$. This set can also be represented graphically as

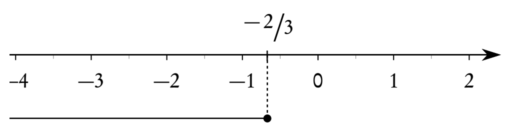


Figure 5.1 *The inequality* $3x + 2 \le 0$

Example 5.3. $2x+8<7x-1$: $-5x<-9$, $5x>9$, $x>9/5$, or $x\in]9/5,+\infty[$. This set can also be represented graphically as

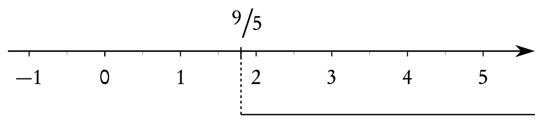


Figure 5.2 *The inequality* $2x+8<7x-1$

Note 5.1. In Example (5.1), the point $-2/3$ is included into the set of solutions. In the second one, instead, the point $9/5$ is not included. Usually a (small) filled circle is used to represent the first kind of points (it is possible to use other conventions. The important thing is to be clear and coherent.).

5.1.2 First order two-variable inequalities

A first order two-variable inequality always assumes one of the following forms

$$ax+by+c>0,\quad ax+by+c\geq 0,\quad ax+by+c<0,\quad ax+by+c\leq 0. \tag{5.2}$$

Equation $ax+by+c=0$ represents a line into the Cartesian plane. In particular, a line divides the plane in two half-planes. So, a first order two-variables inequality has as solution all the points in one of the two half-planes. The points of the line belong to the set of solutions if the sign $=$ appears into the inequality. In order to know which of the two half-planes has to be selected, it is sufficient to take a point (*not belonging to the line*) in one of the two half-planes and check, numerically, if this point satisfies the inequality.

Example 5.4. $2x-y+1>0$. To solve the inequality the following steps are necessary: i) draw the line $2x-y+1=0$, ii) take the point $(0,0)$ (note that $2\times 0-0+1\neq 0$) and substitute its coordinates into the initial inequality. In particular, $(0,0)$ satisfies the inequality. Than, the set of solutions is given by the half-plane determined by the line $2x-y+1=0$ and containing the point $(0,0)$.

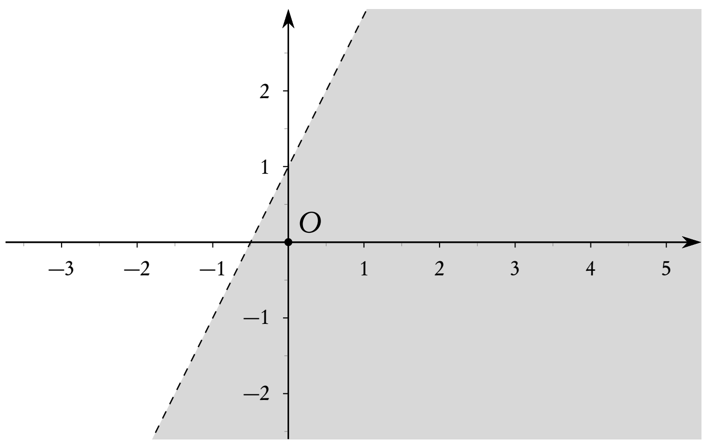


Figure 5.3 *The inequality* $2x-y+1>0$

Example 5.5. $2x+y+1\geq 0$. The set of solutions is represented by the following Figure 5.4:

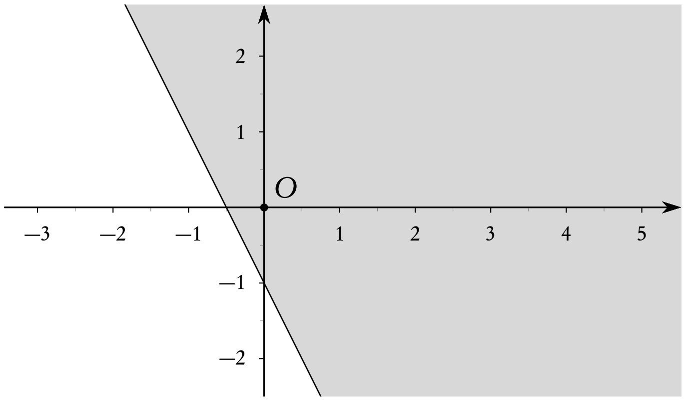


Figure 5.4 *The inequality* $2x+y+1\geq 0$

5.2 Inequalities of second order

5.2.1 One-variable second order inequalities

A second order one-variable inequality has the following representation

$$ax^2+bx+c \lesseqgtr 0. \tag{5.3}$$

The resolution method is very similar to the first-order case[(1)]. Precisely, one first considers and draws the parabola $y=ax^2+bx+c$, then: i) if the sign of the inequality is $\geq$ the range of solutions is given by the xs corresponding to the parts of the parabola which are above the x-axis; ii) if the sign of the inequality is $\leq$ the range of solutions is given by the xs corresponding to the parts of the parabola which are below the x-axis. The following examples will clarify the methodology.

Example 5.6. $2x^2-x-1\geq 0$. Drawing the parabola $2x^2-x-1$ (see Figure 5.5), it is evident that the range of solutions of $2x^2-x-1\geq 0$ are $x\leq -1/2$ or $x\geq 1$, that is

$$x\in \left]-\infty,-\frac{1}{2}\right]\cup[1,+\infty[.$$

[1]Actually there is a method that requires the study of the Δ of the quadratic equation $ax^2+bx+c=0$, associated to the inequality.

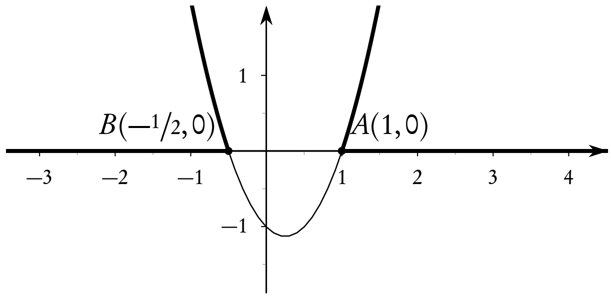


Figure 5.5 *The inequality* $2x^2 - x - 1 \geq 0$

The range of solutions can be represented, graphically, as follows

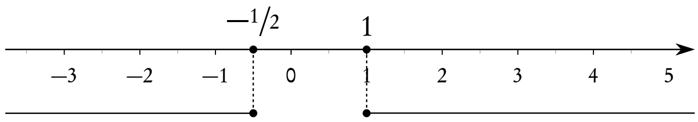


Figure 5.6 *The range of solutions of the ineuality* $2x^2 - x - 1 \geq 0$

Example 5.7. $-2x^2 + x - 1 \geq 0$. Drawing the parabola $-2x^2 + x - 1$ (see Figure 5.7), it is evident that the inequality does not admit any solution. Instead, the inequality $-2x^2 + x - 1 \leq 0$ (resp. $-2x^2 + x - 1 < 0$) has as solution the set $\mathbb{R}$ (resp. the set $\mathbb{R}$ deprived from the element 0).

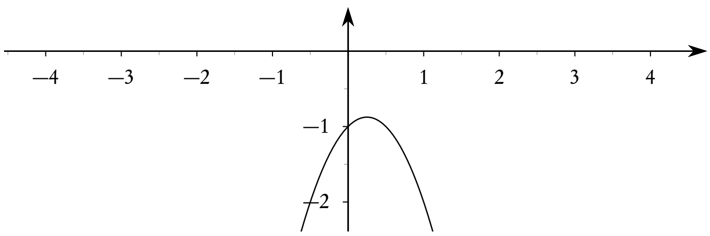


Figure 5.7 *The inequality* $-2x^2 + x - 1 \geq 0$.

Example 5.8. $x^2 + 2x + 1 \leq 0$. Drawing the parabola $x^2 + 2x + 1 = 0$ (see Figure 5.8), it is evident that range of solutions of the inequality is given only by the point $x = -1$. Indeed, the polynomial $x^2 + 2x + 1 = (x + 1)^2$ is always positive except at $x = -1$.

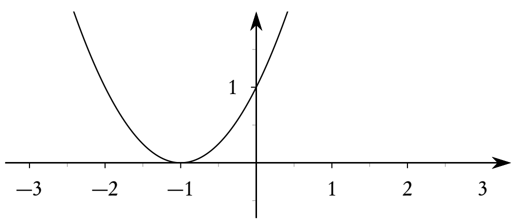


Figure 5.8 *The inequality* $x^2 + 2x + 1 \leq 0$

5.2.2 Two-variable second-order inequalities

The resolution method is analogous to the case of the two-variable first order inequalities and the following examples will clarify the procedure.

Example 5.9. $x^2+y^2-2x-2y+1\leq 0$. To solve this inequality the following steps are necessary: i) draw the circumference of equation $x^2+y^2-2x-2y+1=0$, ii) check if an internal point (for example the center) satisfies the inequality, then iii) if the center satisfies the inequality, the set of solutions is given by all the internal points and the points of the circumference (observe that the inequality is of type "$\leq$").

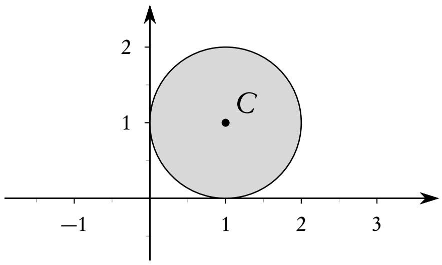


Figure 5.9 *The inequality* $x^2+y^2-2x-2y+1\leq 0$

Example 5.10. $x^2-\dfrac{y^2}{4}<1$. Following the same procedure above and trying with the point $(0,0)$, it is possible to conclude that the inequality is verified by all the points between the two branches of the hyperbola $x^2-\dfrac{y^2}{4}=1$. However, in this case the sign of the inequality is $<$.

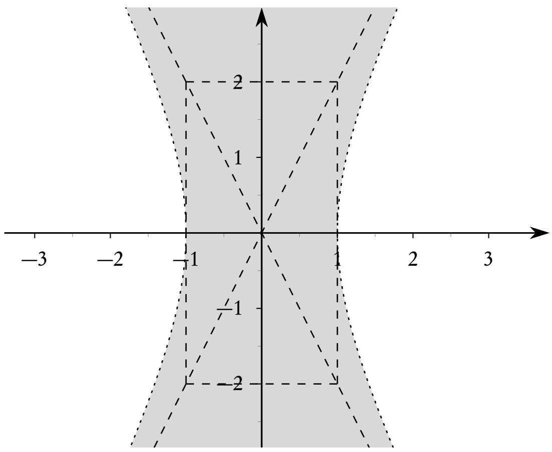


Figure 5.10 *The inequality* $x^2-\dfrac{y^2}{4}<1$

5.3 Systems of inequalities

A system of inequalities is a set of inequalities in the same variables. Similarly to the systems of equations case, a *solution* to a system of inequalities is an assignment of numbers to the variables such that all the inequalities are simultaneously satisfied. In this case, however, the graphical representation will help significantly.

5.3.1 One-variable systems of inequalities

Example 5.11. $\begin{cases} 2x-1 \le 0 \\ x^2-5x+4>0 \end{cases}$. To solve the system it is necessary to solve separately each inequality. The first has as solution $x \le 1/2$, while the second $x<1 \vee x>4$. Drawing the graph in Figure 5.11, it is easily to deduce that the solutions of the system are given by

$$\left]-\infty, \frac{1}{2}\right].$$

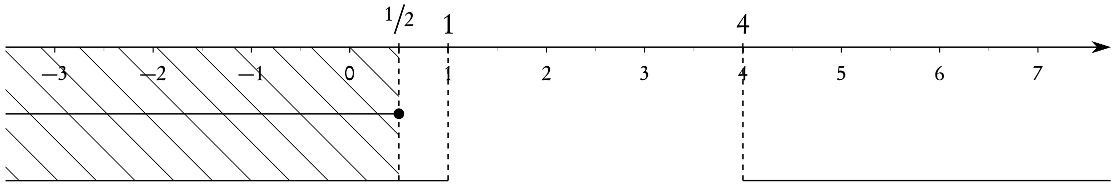


Figure 5.11 *Graphical representation of a one-variable system of inequalities*

5.3.2 Two-variable systems of inequalities

The method of resolution is very similar to that presented in the previous section. One first solves, separately, each inequality of the system then one intersects the solutions. Also in this case, the graphical representation will be essential. We present the following example

Example 5.12. $\begin{cases} x^2+y^2-2x>0 \\ x-y-2>0 \end{cases}$.

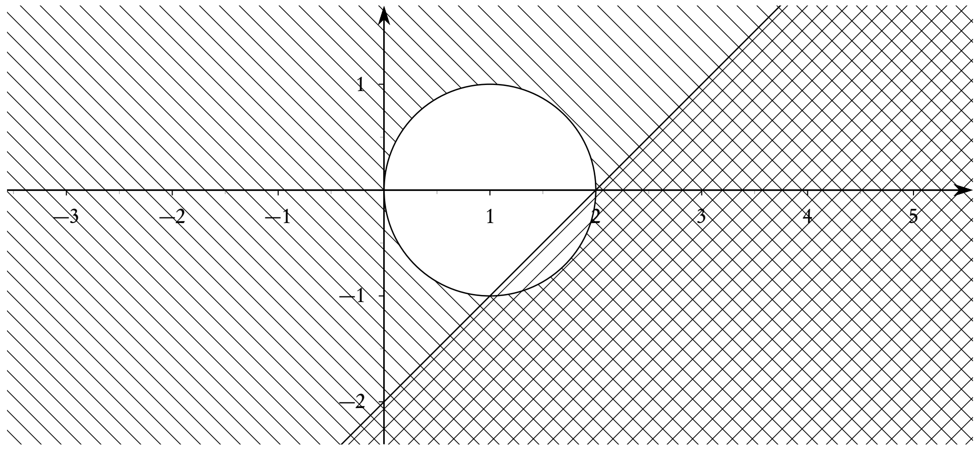


Figure 5.12 *Graphical representation of a two-variable system of inequalities*

The solution set is given by the chequered plane.

5.4 Factorable polynomial inequalities

Let suppose to have a one-variable or a two-variable inequality given by

$$f(x) \lesseqgtr 0, \quad f(x,y) \lesseqgtr 0.$$

and that the quantities $f(x)$ or $f(x,y)$ are not a first- or a second-order polynomials. In this case, the so called *rule of signs* is used to solve the inequality. Basically the rule of signs states that the product of two numbers with the same signs (resp. different signs) is positive (resp. negative). So, if the polynomial $f(x)$ or $f(x,y)$ is a factorable polynomial, in order to solve the inequality, one has first to determine the sign of each factor, then of the product. To facilitate the analysis, a graphical representation is used. The following examples will clarify the concepts.

Example 5.13. $(x^2-1)(x-2)>0$. The polynomial $f(x)=(x^2-1)(x-2)>0$ is a factorable polynomial with factors (x^2-1) and $(x-2)$. In particular, the first factor, x^2-1, is positive for $x<-1$ and $x>1$, negative for $-1<x<1$, and zero for $x=\pm1$ ($y=x^2-1$ is the equation of a parabola). The second factor, $x-2$ is positive for $x>2$, negative for $x<2$, and zero for $x=2$. The conclusions are easily stated if we consider the graph in the following Figure 5.13

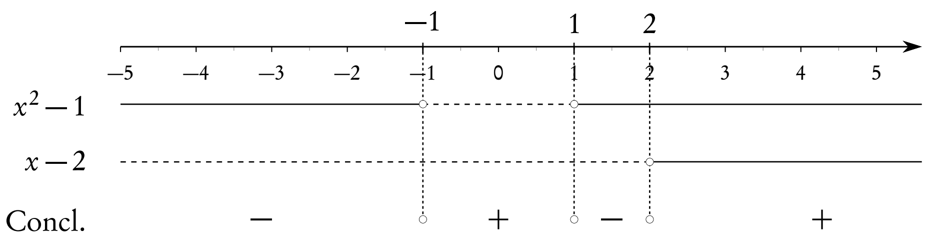


Figure 5.13 *Graphical representation of the sign of the inequality* $(x^2-1)(x-2)$

Note that

- a plain line is used to indicate the parts where each factor is positive ;
- a dotted line is used to indicate the parts where each factor is negative ;
- a 0 is used to indicate the points where each factor is exactly equal to zero .

Summarising, the inequality is satisfied for

$$x \in]-1,1[\cup]2,+\infty[.$$

Note that to solve the inequality $(x^2-1)(x-2)<0$ it is not necessary to construct a new graph. In particular, $(x^2-1)(x-2)<0$ is satisfied for

$$x \in]-\infty,-1[\cup]1,2[.$$

Analogously, the inequalities $(x^2-1)(x-2) \le 0$ and $(x^2-1)(x-2) \ge 0$ has as solutions

$$]-\infty,-1] \cup [1,2],$$

and

$$[-1,1] \cup [2,+\infty[,$$

respectively.

Example 5.14. $\dfrac{x^2-1}{x-2} \ge 0$. To solve this inequality the rule of signs is again used (the rule of signs is valid also for the quotient between two numbers). The main difference is that, in this case, the factor $x-2$ is at the denominator of the fraction and hence, it has to be different from zero. In what follows, the symbol $\times$ is used to indicate that, for example, the value $x=2$ cannot belong to the set of solutions. Figure 5.14 represents, graphically, the inequality:

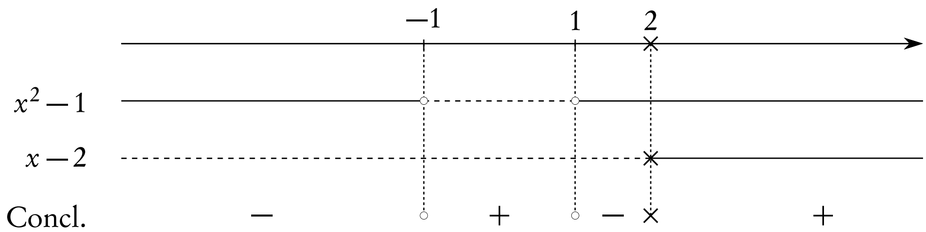


Figure 5.14 *Graphical representation of the sign of the inequality* $(x^2-1)/(x-2) \ge 0$

The inequality is satisfied for

$$x \in [-1,1] \cup]2,+\infty[.$$

Example 5.15. $\dfrac{x-y+1}{x^2+y^2-2y} \ge 0$. The first step consists in determining the sign of the numerator and denominator separately. Then, using the rule of signs one determines the sign of the quotient. In this case, the range of solutions will be a subset of the plane. Figures 5.15 and 5.16 represent the set of positivity of the numerator and of the denominator, respectively. In particular: i) the gray regions indicate the sets of positivity, i) the white regions indicate the sets of negativity, iii) the line and the circle indicate the points in which the numerator and the denominator are equal to zero; these points have to be excluded. Figure 5.17 represents the set of solutions of the inequality. Note that we have to exclude the entire circumference and the points of intersection between the line and the circumference.

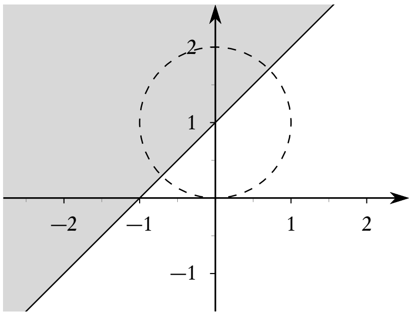


Figure 5.15 $x-y+1>0$

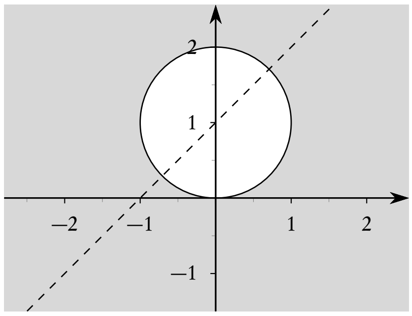


Figure 5.16 $x^2+y^2-2y>0$

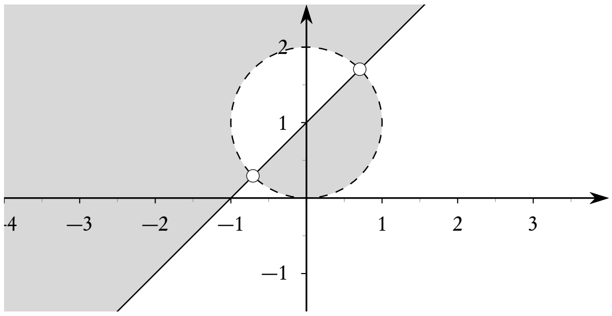


Figure 5.17 $\dfrac{x-y+1}{x^2+y^2-2y}\geq 0$

5.5 Inequalities with radicals

In this course only two types of *inequalities with radicals* are considered. In particular:

1. $\sqrt{f(x)}\geq g(x)$ (or $\sqrt{f(x)}>g(x)$);
2. $\sqrt{f(x)}\leq g(x)$ (or $\sqrt{f(x)}<g(x)$).

To solve the first type, one has to consider the union of the solutions of the two systems

$$\begin{cases} f(x)\geq 0 \\ g(x)<0 \end{cases} \cup \begin{cases} f(x)\geq g^2(x) \\ g(x)\geq 0 \end{cases}, \quad \left(\text{or} \begin{cases} f(x)\geq 0 \\ g(x)<0 \end{cases} \cup \begin{cases} f(x)>g^2(x) \\ g(x)\geq 0 \end{cases}\right).$$

To solve the second type, instead, one has to consider the union of the solutions of the two systems:

$$\begin{cases} f(x)\geq 0 \\ g(x)\geq 0 \\ f^2(x)\leq g(x) \end{cases}, \quad \left(\text{or} \begin{cases} f(x)\geq 0 \\ g(x)\geq 0 \\ f^2(x)<g(x) \end{cases}\right).$$

Example 5.16. $\sqrt{x^2-9x+14}>x-8$.

$$\begin{cases} x^2-9x+14\geq 0 \\ x-8<0 \end{cases} \cup \begin{cases} x^2-9x+14>(x-8)^2 \\ x-8\geq 0 \end{cases}.$$

The first system is satisfied by $x \le 2 \vee 7 \le x < 8$; the second system is satisfied by $x \ge 8$. So, the range of x values such that $x \le 2 \vee x \ge 7$ is the solution of $\sqrt{x^2-9x+14} > x-8$.

Example 5.17. $\sqrt{4x^2-13x+3} < 2x-3$.

$$\begin{cases} 4x^2-13x+3 \ge 0 \\ 2x-3 \ge 0 \\ 4x^2-13x+3 < (2x-3)^2 \end{cases}.$$

Exercise

To solve a radical inequality containing only a radical with odd index root (in particular, with odd index root equal to 3) it is sufficient to cube both members of the inequality and then, to solve the new inequality.

Example 5.18. $\sqrt[3]{x^2+7} > 2$. If we cube both members, we obtain $x^2-1>0$, which has as solutions $x < -1 \vee x > 1$.

5.6 Exercises

Exercice 5.1. *Solve the following inequalities.*

1. $x^2+3x+2>0$;
2. $-x^2-3x+2<0$;
3. $4-x^2>0$;
4. $x^2-x+6<0$;
5. $(x^2+2x-8)(x+1)>0$;
6. $(x^2-2)(x+1)(1-x) \ge 0$;
7. $x(x^2+2)(2x-1)<0$;
8. $\dfrac{x+1}{x^2+1}<0$;
9. $\dfrac{2x-8}{1-x-x^2}>0$;
10. $\dfrac{x^2-4}{x+3} \le 0$;
11. $x^3-27 \ge 0$;
12. $2-x^3<0$;
13. $x^3(x^2-1)(2-x^2) \le 0$;
14. $\dfrac{x-9}{x^3+1} \ge 0$;
15. $\dfrac{8-x^3}{x^3+9} \le 0$.

Exercice 5.2. *Solve the following systems of inequalities.*

1. $\begin{cases} x^2-1>0 \\ 2x+3\geq 0 \end{cases};$
2. $\begin{cases} x+1>0 \\ x^2+2x-8>0 \end{cases};$
3. $\begin{cases} x+1<0 \\ x^2+1<0 \end{cases};$
4. $\begin{cases} 2x-8>0 \\ 1-x-x^2<0 \end{cases};$
5. $\begin{cases} (1-3x^2)(x-2)<0 \\ (2+x)(1-x)>0 \end{cases};$
6. $\begin{cases} 3x-2<0 \\ 2x(3-x)>0 \end{cases};$
7. $\begin{cases} \dfrac{3}{x}<0 \\ \dfrac{2x+1}{x(2-3x)}>0 \end{cases};$
8. $\begin{cases} \dfrac{x-3}{x}<0 \\ \dfrac{x+1}{1-x}>0 \end{cases}.$

Exercice 5.3. *Solve the following systems of inequalities.*

1. $\sqrt[3]{\dfrac{1}{1-x}}<1;$
2. $\sqrt{\dfrac{x^3}{x-1}}>x+1;$
3. $\sqrt{1-x^2}<1-x;$
4. $\sqrt{x}<x;$
5. $\sqrt{1-x^2}>x^2;$
6. $\sqrt{x(x+1)}<1-x.$

Exercice 5.4. *Determine, graphically, the solutions of the following systems of inequalities.*

1. $\begin{cases} y-x+1>0 \\ 2x-3\leq 0 \end{cases};$

2. $\begin{cases} x+2y-1>0 \\ 2x+3y+2\geq 0 \end{cases}$;

3. $\begin{cases} x+1\geq 0 \\ y+1>0 \end{cases}$;

4. $\begin{cases} x-8>0 \\ 1-x<0 \end{cases}$;

5. $\begin{cases} y-1>0 \\ y+3<0 \end{cases}$;

6. $\begin{cases} x+y-1>0 \\ x-2y\leq 0 \\ x+y<0 \end{cases}$;

7. $\begin{cases} x^2+y^2-1<0 \\ x-y\leq 0 \\ x+3y>0 \end{cases}$;

8. $\begin{cases} (x-1)^2+y^2-4>0 \\ x+y+2\leq 0 \\ x-2y-1<0 \end{cases}$;

9. $\begin{cases} (x-1)^2+(y-2)^2-9<0 \\ y-x+2\leq 0 \\ x+1>0 \end{cases}$;

10. $\begin{cases} x^2+y^2-4>0 \\ (x-1)^2+(y-1)^2\leq 4 \\ y-2x<0 \end{cases}$;

11. $\begin{cases} (x-2)^2+(y-1)^2-1<0 \\ x^2+(y-2)^2\leq 40 \\ x+y<0 \end{cases}$;

6 Exponentials and Logarithms

6.1 Powers

If a is any real number and m is a natural number *greater or equal than* 2, the *power* of *base a* and *exponent m* is given by the following number

$$a^m = \underbrace{a \cdot a \cdots a}_{m\text{-times}}. \tag{6.1}$$

If $m = 1$ and a is any real number, it is set, by definition,

$$a^1 = a\,. \tag{6.2}$$

Note that a^1 *is not* a product: *Two* factors are needed to compute a product.
If a is any real number *different* from zero, it is set, by definition

$$a^0 = 1\,. \tag{6.3}$$

The expression 0^0 *has no* meaning. The power of base any real number a, *different from zero*, and exponent a negative integer number is given by

$$a^{-m} = \frac{1}{a^m}, \quad a \neq 0. \tag{6.4}$$

Similarly to the Formula (6.3), the symbol $0^{\text{negative.number}}$ *has no* meaning. It is also possible to define the power of base a and exponent any real number. In this case, the base a has to be *greater or equal than zero* if the exponent is not negative. If, instead, the exponent is a rational number of type m/n, with n a natural number greater than > 1, it is set, by definition,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, \quad a > 0; \qquad 0^{\frac{m}{n}} = 0, \quad \frac{m}{n} > 0\,. \tag{6.5}$$

The extension to the case of power of any real exponent (for example $a^{\sqrt{2}}$) is more complex and it is beyond the purposes of this crash introduction. However, let describe the method in a specific case. For instance, let suppose that the problem is to compute $a^{\sqrt{2}}$. In this case, it is necessary to consider the successive decimal approximation of $\sqrt{2}$ with an increasing number of decimal digits. In particular

$$1.4 = \frac{14}{10}, \quad 1.41 = \frac{141}{100}, \quad 1.414 = \frac{1414}{1000}, \quad 1.4142 = \frac{14142}{10000}, \quad \ldots$$

It is possible to compute a raised to each of the exponents that approximate $\sqrt{2}$ because they are rational numbers. So, $a^{\sqrt{2}}$ will be the *limit value* of this sequence of numbers when the exponent tends to $\sqrt{2}$.

Finally, the following properties hold (remind that the base has to be positive if the exponent is not an integer number, and different from zero if it appears in the denominator of a fraction)

$$(a^m)^n = a^{mn}; \tag{6.6}$$

$$a^m \cdot a^n = a^{m+n}; \tag{6.7}$$

$$\frac{a^m}{a^n} = a^{m-n}. \tag{6.8}$$

6.2 Power functions

In mathematics, a *power function* is a function of the form

$$f(x) = x^a, \tag{6.9}$$

where a is constant and x is a variable. In general, a can belong to one of several classes of numbers, such as $\mathbb{Z}$. The domain of a power function is $\mathbb{R}$ when a is a positive integer and $\mathbb{R}$ deprived from the element zero when a is a negative integer. In all the other cases the domain is $\mathbb{R}^+$. It is important to note that Equation (6.9) represents the equation of a line passing through the origin $(0,0)$ and with slope 1 (i.e. the bisector of the first and the third quadrant) if $a = 1$. When $a = 2$, instead, it represents the equation of a parabola with vertex at the origin and upward concavity. Figures below represent these functions together with other examples of power functions.

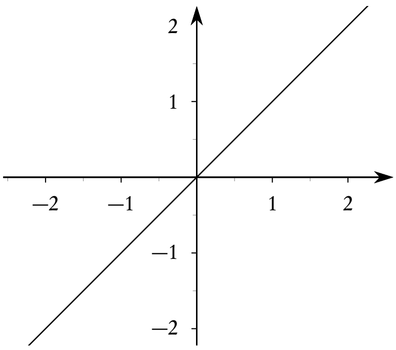


Figure 6.1 *Graph of the function* $f(x) = x^1 = x$

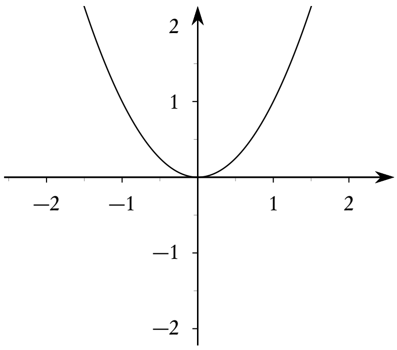


Figure 6.2 *Graph of the function* $f(x) = x^2$

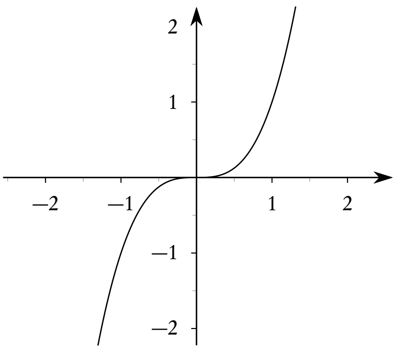


Figure 6.3 *Graph of the function* $f(x) = x^3$

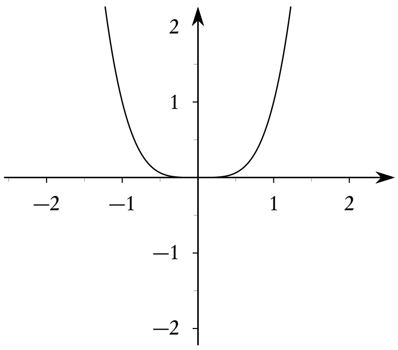


Figure 6.4 *Graph of the function* $f(x) = x^4$

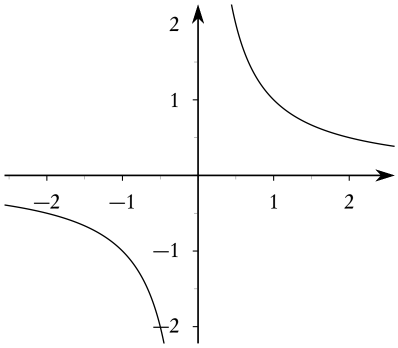


Figure 6.5 *Graph of the function* $f(x) = x^{-1} = 1/x$

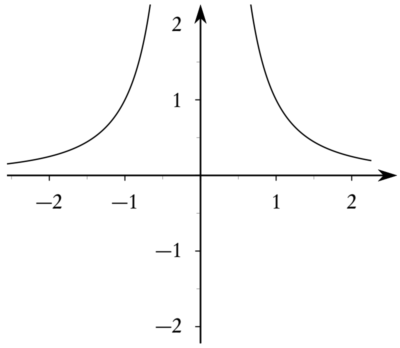


Figure 6.6 *Graph of the function* $f(x) = x^{-2} = 1/x^2$

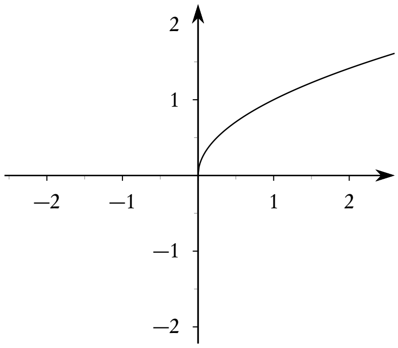


Figure 6.7 *Graph of the function* $f(x) = x^{1/2} = \sqrt{x}$

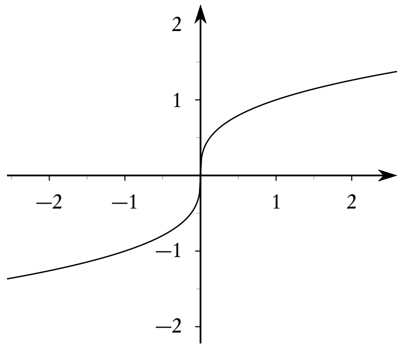


Figure 6.8 *Graph of the function* $f(x) = \sqrt[3]{x}$

Note 6.1. The function $x^{1/3}$ is different from the function $\sqrt[3]{x}$. In particular, the first is defined for $x \geq 0$, whereas the second $\forall x \in \mathbb{R}$. Besides, the graph of x^a passes through the point $(1,1)$ whichever is the value of the exponent a.

Together with power functions, *exponential* and *logarithmic functions* are bricks used for building a considerable amount of mathematical models in many applications.

6.3 Exponential function

In mathematics, an *exponential function* is a function of the form

$$f(x) = a^x, \quad a > 0, \tag{6.10}$$

where a, the *base*, is a positive real number different from 1 (the case $a = 1$ is not interesting because $1^x = 1$, $\forall x \in \mathbb{R}$). The following figures report some examples of exponential functions.

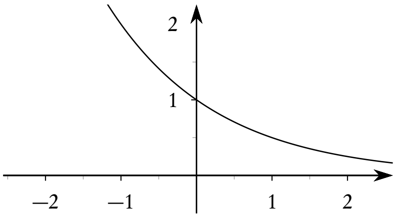


Figure 6.9 *Graph of the function* $f(x) = (1/2)^x$

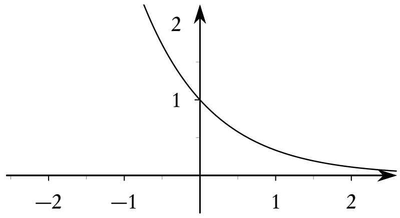


Figure 6.10 *Graph of the function* $f(x) = (1/3)^x$

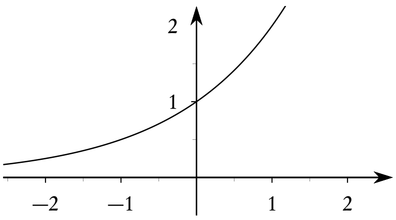


Figure 6.11 *Graph of the function* $f(x) = 2^x$

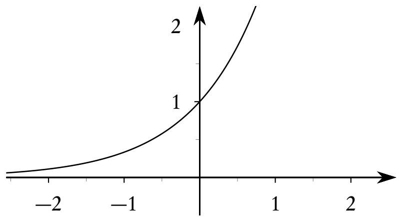


Figure 6.12 *Graph of the function* $f(x) = 3^x$

Exponential functions are positive for each value of a and their graph passes through the point $(0,1)$. A glance at Figures 6.9, 6.10, 6.11, 6.12 reveals that they are

1. Strictly increasing if $a > 1$.
2. Strictly decreasing if $0 < a < 1$.

Besides, when a is greater than 1, the rate of growth of exponential functions is very large. Table 6.1 clarifies this fact with an example.

x	x^2	2^x
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
10	100	1024
100	10000	$\sim 1.27 \cdot 10^{30}$

Table 6.1 *Comparison between* x^2 *and* 2^x

There is a privileged base for exponential functions: The irrational *Napier's number* e. Its value is approximatively

$$e \simeq 2.718 .$$

Because $e > 1$, the corresponding exponential function is increasing. In what follows, if not explicitly specified, the base e is always used and the related exponential function is denoted by $\exp(x)$.

6.4 Logarithmic functions

In mathematics, a general problem is to determine the solution of exponential equations of type $2^x = 8$. It is evident that this equation has as unique solution the value $x = 3$. This fact is also confirmed, graphically, by the following Figure 6.13.

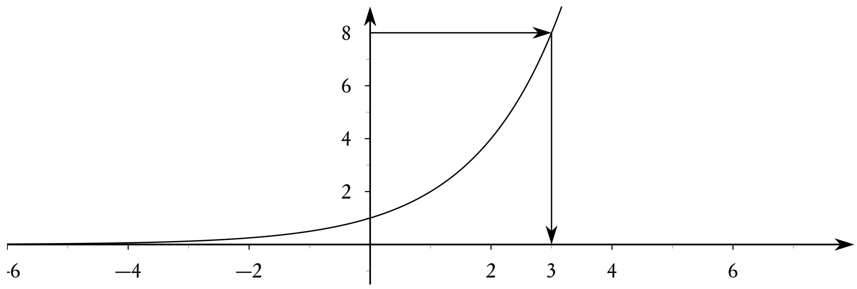


Figure 6.13 *Graphical representation of equation* $2^x = 8$

Instead, finding the solution of $2^x = 3$ is not so trivial. Observing Figure 6.14, it is evident that a solution exists.

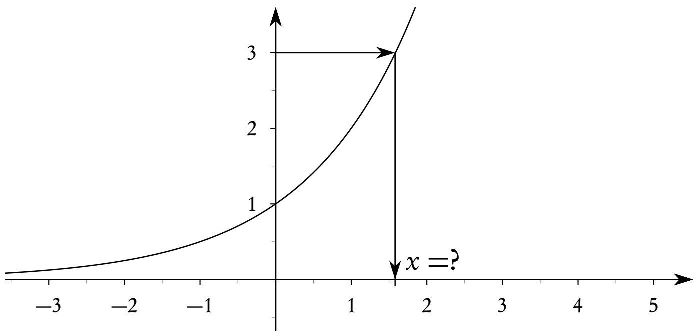


Figure 6.14 *Graphical representation of equation* $2^x = 3$

The concept of logarithm is introduced for solving an equation like $2^x = 3$.

Definition 6.1. *Let a a real number,* $a > 0$ *and* $a \neq 1$. *The* logarithm with base a of b *is defined as the* exponent *assigned to the base a in order to obtain* b. *In symbols*

$$\log_a(b), \quad \textit{or simply} \quad \log_a b\,. \tag{6.11}$$

The previous definition can be summarised as

$$a^{\log_a b} = b\,. \tag{6.12}$$

Using this definition the solution of $2^x = 3$ is given by $\log_2 3$ because

$$2^{\log_2 3} = 3\,.$$

Example 6.1. $\log_3 81 = 4$, because $3^4 = 81$.

Example 6.2. $\log_{10} 1000 = 3$, because $10^3 = 1000$.

Example 6.3. $\log_2 \dfrac{1}{16} = -4$, because $2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{16}$.

Example 6.4. $\log_{10} \dfrac{1}{10} = -1$, because $10^{-1} = \dfrac{1}{10}$.

Also for logarithmic functions, the most important base is the Napier's number e. In what follows, to refer to logarithms with base e, the symbol ln (instead of $\log_e$), which stands for *natural logarithm*, is used. *This notation is not universal. Sometimes, the natural logarithm is also denoted by* $\log x$. *In this course, instead, the notation* $\log x$ *is used to indicate the logarithm to the base* 10 *of* x.:

$$\log_e x = \ln x \,. \tag{6.13}$$

The following properties hold

$$\log_a(xy) = \log_a x + \log_a y\,, \quad x > 0,\ y > 0; \tag{6.14}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y\,, \quad x > 0,\ y > 0; \tag{6.15}$$

$$\log_a (x)^y = y \log_a x\,, \quad x > 0; \tag{6.16}$$

$$\log_a a = 1\,; \tag{6.17}$$

$$\log_a 1 = 0\,. \tag{6.18}$$

In particular, previous properties together with Formula (6.12) imply the following relations

$$a^{\log_a x} = x,\ \forall x > 0, \qquad \log_a a^x = x,\ \forall x \in \mathbb{R}. \tag{6.19}$$

These relations indicate that the *inverse* of the logarithmic function is the exponential function and, conversely, the inverse of the exponential function is the logarithmic function.

Figures 6.15 and 6.16 report the graphics of the logarithmic function with base greater (Figure 6.15) and smaller (Figure 6.16) than 1, respectively. In particular, the same observations on monotonicity done for exponential functions are valid.

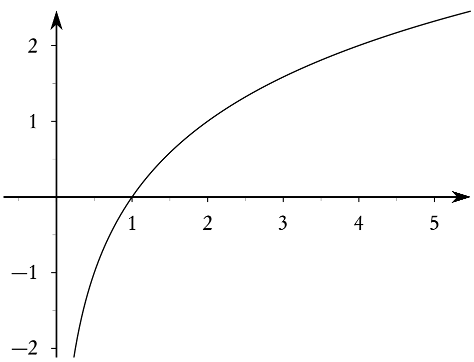


Figure 6.15 *Graph of the function* $\log_2 x$

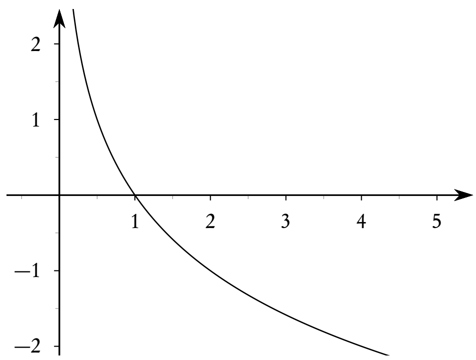


Figure 6.16 *Graph of the function* $\log_{1/2} x$

All logarithmic functions pass through the point $(1,0)$ and have as domain the set of all the real numbers *strictly* greater than zero.

Finally, there is one other log rule. Actually, it is more of a formula than a rule. You may have noticed that your calculator only has keys for figuring the values for the common logarithm with base 10 or base e, but no other bases. In order to evaluate a logarithm with a non-standard-base, the *change-of-base-formula* is used. It reads as

$$\log_a b = \frac{\ln b}{\ln a}\,. \tag{6.20}$$

6.5 Exponential and logarithmic inequalities

In this section, only some examples of exponential and logarithmic inequalities are considered.

Example 6.5. $2^x > 32\,(=2^5)$. First observations: The base (2) is greater than 1 and so the exponential function is strictly increasing.
It is sufficient to remind power properties to conclude that the solution is $x > 5$.

Example 6.6. $3^x < 5$. First observations: The base (3) is greater than 1 and so the exponential function is strictly increasing.
To solve the inequality, one has to apply the natural logarithm on both sides, $\ln 3^x < \ln 5$, obtaining $x \ln 3 < \ln 5$ and, finally, $x < \dfrac{\ln 5}{\ln 3}$.

Example 6.7. $2^{x^2-1} > 8$. First observations: The base (3) is greater than 1 and so the exponential function is strictly increasing.
Second observation: $8 = 2^3$ so $2^{x^2-1} > 8$ is equivalent to $2^{x^2-1} > 2^3$. So, $2^{x^2-1} > 2^3$, from which $x^2-1 > 3$, $x^2-4 > 0$ and, finally, $x < -2 \vee x > 2$.

Example 6.8. $\ln(2x^2+x) > 0$. First observations: The domain of the logarithmic function is $\mathbb{R}_+$. In particular, the condition $2x^2+x > 0$, or equivalently $x < -1/2 \vee x > 0$, has to be imposed.
To solve $\ln(2x^2+x) > 0$, one has to apply the exponential with base e on both sides

$$e^{\ln(2x^2+x)} > e^0, \Rightarrow 2x^2 - x > 1, \Rightarrow 2x^2 - x - 1 > 0, \Rightarrow x < -1 \vee x > 1/2\,.$$

Matching the solution with the existence condition $x < -1/2 \vee x > 0$, it is possible to conclude that the inequality is satisfied for $x < -1 \vee x > 1/2$.

The following examples are left as an exercise.

Example 6.9. $\ln(x-1) \geq \ln(-x+3)$.

Example 6.10. $2^x > -3$.

7 Trigonometry

7.1 Angles and radiants

In planar geometry, an *angle* is the figure formed by two rays, called the *sides* of the angle, sharing a common endpoint, called *vertex*. Angles are measured in *grades* or *radiants*. An angle of one grade, denoted by 1°, corresponds to the 360^{th} part of the round angle. The right angle measures 90°.

However, in mathematical analysis another method is used to measure angles. Let $A\hat{O}B$ an angle with vertex O and let AB the arc individuated by the circumference with center O and radius r. The ratio between the length a of the arc AB and the measure of the radius r is the *measure in radiants* of the angle $A\hat{O}B$. This measure is a *pure number* because it is a ratio between two quantities having the same unit of measure. If $r = 1$ then the length of the arc is equal to the length of the angle (see Figure 7.1).

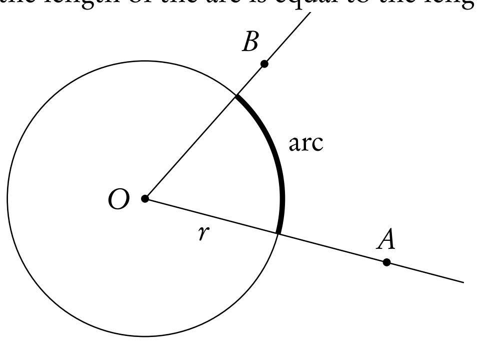


Figure 7.1 *Angles and their measure in radiants*

Table 7.1 reports some of the most used measures of angles (symbol $\alpha°$ is used to indicate the measure in grade whereas α to indicate the measure in radiants).

$\alpha°$	α
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π
270°	$3\pi/2$
360°	2π

Table 7.1 *Angles and their measure in grades and radiants*

Although the definition of the measurement of an angle does not support the concept of negative angle, in applications, it is frequently useful to impose a convention that allows positive and negative angular values, to represent orientations and/or rotations in opposite directions relative to some reference. A positive sign is attributed to angles oriented anti-clockwise and negative to angles oriented clockwise.
There exists a direct correspondence between circumference arcs and angles. To measure an angle one moves (clockwise or anti-clock-wise) from the point of intersection between the circumference and the first side to the point of intersection between the circumference and the second side. In particular, it is possible to "travel" the circumference more than one time, obtaining angles "larger" than 2π. These angles are named *generalized angles*.
For example, with reference to Figure 7.2, you can imagine to start from the point P and "travel" the arc (of length 1) to join the point Q. At this point, you continue along the circumference to reach again the point Q. In this case the length of the "journey" will be $2\pi + 1$.

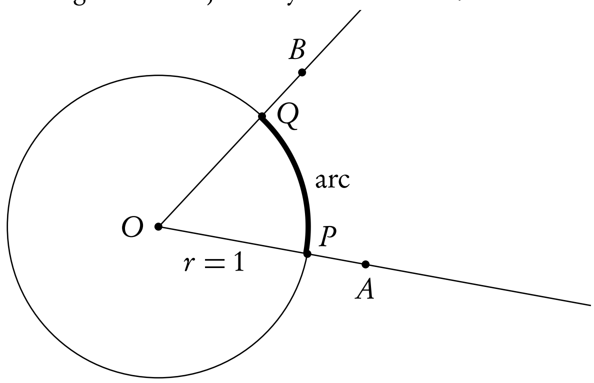


Figure 7.2 *Generalized angles*

When working with a Cartesian plane, it is always possible to assume that the vertex of the angle coincides with the origin and the first side with the positive semi-positive x-axis. In this situation, to measure angles, it is necessary to draw the circumference described by equation $x^2 + y^2 = 1$. This circumference, with unit radius, is named *geometric circumference*. At this point, angles are identified with the arcs of this circumference. Moreover, it is possible to associate with any real number a point on the circumference (this association is *not* unique because of the definition of generalized angles), by "travelling" the circumference (clockwise or anti-clockwise), starting from the point $(0, 1)$, for an arc of length the absolute value of the real number.

7.2 Sine and cosine functions

Le $P = (x_P, y_P)$ the point on the geometric circumference associated with the real number x. The abscissa, x_P, and the ordinate, y_P, of P have a great impact on applications. In particular, the following definition holds.

Definition 7.1. *The abscissa of the point P is named* cosine *of the real number x; the ordinate of the point P is named* sine *of the real number x. Precisely:*

$$x_P = \cos(x), \quad y_P = \sin(x), \text{ or, simply } x_P = \cos x, \quad y_P = \sin x. \tag{7.1}$$

Figures 7.3 and 7.4 represent the functions introduced above.

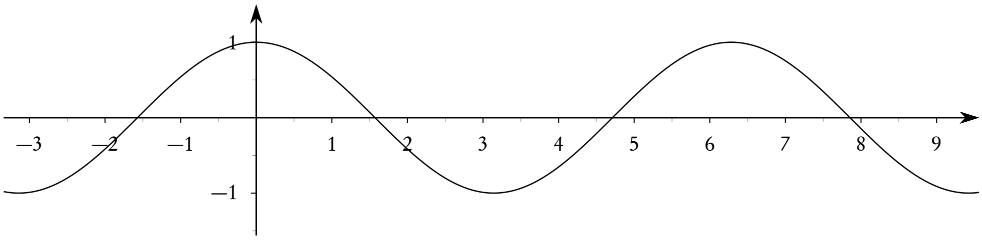


Figure 7.3 *The cosine function*

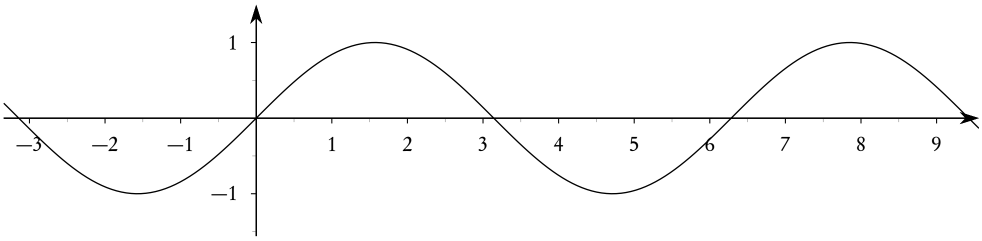


Figure 7.4 *The sine function*

Both sine and cosine functions (hereafter *trigonometric functions*) are *periodic*. In mathematics, a periodic function is a function that repeats its values in regular intervals or periods. Trigonometric functions repeat over intervals of 2π. Periodic functions are used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity. Any function which is not periodic is called *aperiodic*. Figure 7.5 represents a function obtained by opportunely mixing trigonometric functions.

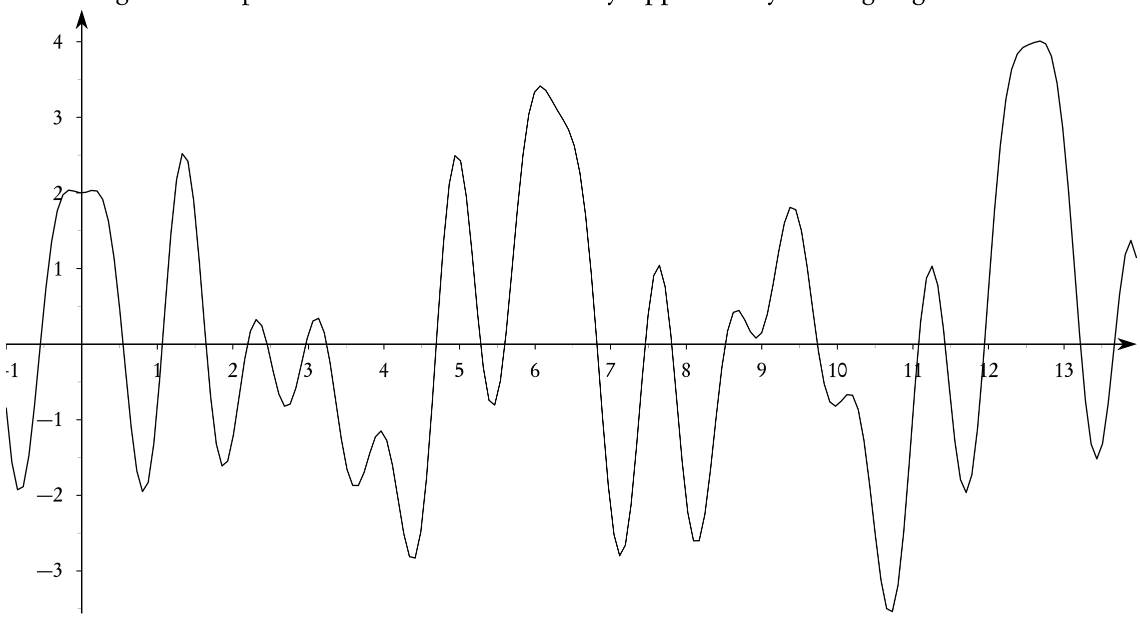


Figure 7.5 *Oscillatory function*

7.3 Addition formulae

In this section some important formulae linked to trigonometric functions are given. In particular, the *sum and difference formulae*.

(7.2) $$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y, \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y\,.$$

For instance, from

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}\,,\ \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\,,\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\,,\ \sin\frac{\pi}{6} = \frac{1}{2}\,,$$

one obtains

$$\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}\,.$$

In particular, setting $x = y$:

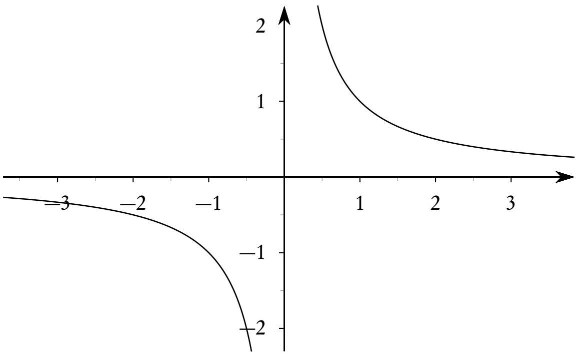
(7.3) $$\cos(2x) = \cos^2 x - \sin^2 x\,, \quad \sin(2x) = 2\sin x \cos x\,.$$

8 Elementary Graph of Functions

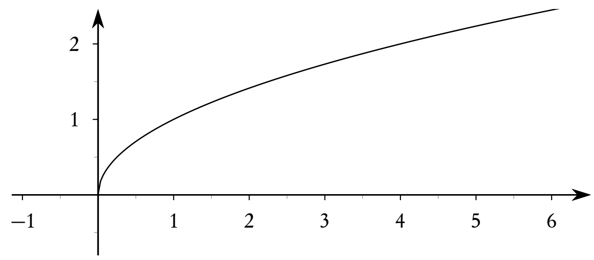
In this chapter we present some basic techniques used to draw graphs of functions. In particular, in Section 8.1 we report some of the graphs that we have seen in previous chapters. In Section 8.2.1, instead, we introduce the *absolute value* function.

8.1 Some graphs of functions

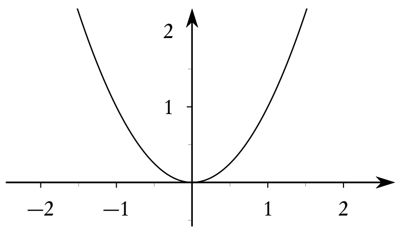
- $f(x) = \dfrac{1}{x}$



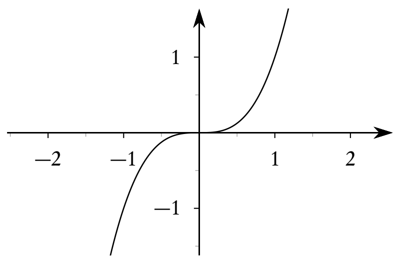
- $f(x) = \sqrt{x}$



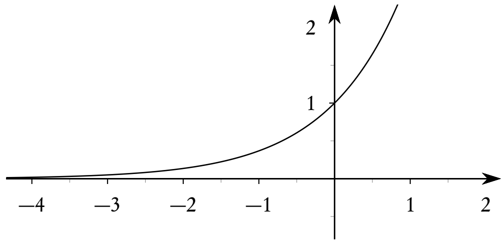
- $f(x) = x^2$



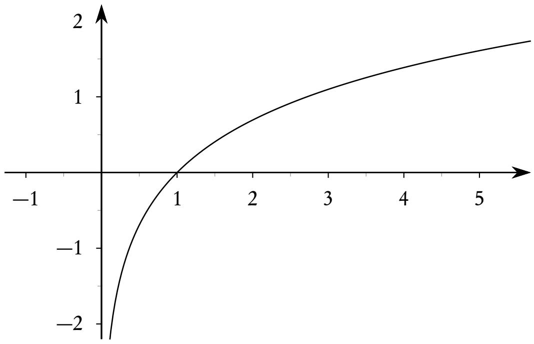
- $f(x) = x^3$



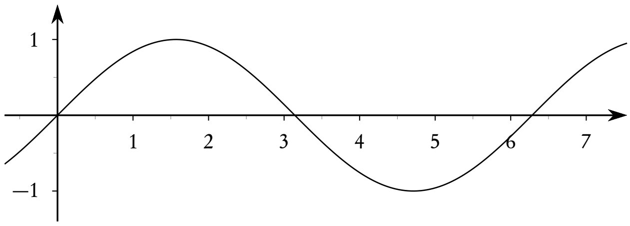
– $f(x) = e^x$



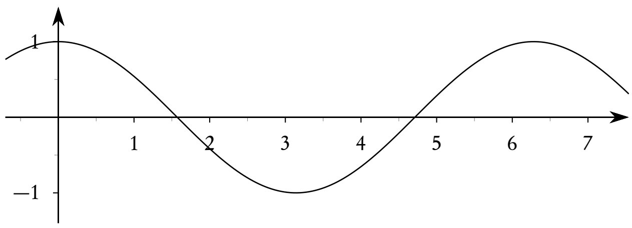
– $f(x) = \ln(x)$



– $f(x) = \sin(x)$



– $f(x) = \cos(x)$



8.2 Absolute value or modulus

8.2.1 Absolute value function

Definition 8.1. *The non-negative real number defined by the formula*

$$|x| = \begin{cases} x, & \textit{if} \quad x \geq 0 \\ -x, & \textit{if} \quad x < 0 \end{cases} . \tag{8.1}$$

is called the absolute value *(or the* modulus*) of* x.

The modulus of x thus represents the *distance* from the origin of the point on the real line with abscissa x. The graph of the absolute value function, $f(x) = |x|$, is reported in Figure 8.1.

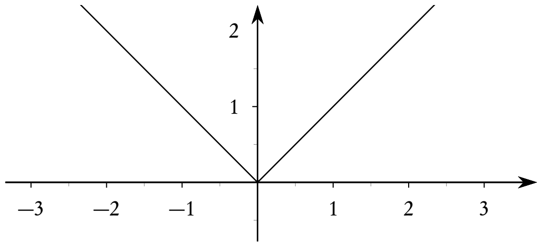


Figure 8.1 *Graph of the* absolute value function

Example 8.1. $|5| = 5$.

Example 8.2. $|-3| = 3$.

Example 8.3. $\left|1-\sqrt{2}\right| = -(1-\sqrt{2}) = \sqrt{2}-1$.

Example 8.4. $|-2| + \left|\sqrt{5}-2\right| = 2 + (\sqrt{5}-2) = \sqrt{5}$.

8.2.2 Properties of the absolute value function

The absolute value function satisfies the following properties

- $|x| \geq 0$, $\forall x \in \mathbb{R}$.
- $|x| = 0$ if and only if $x = 0$.
- $|x| = |-x|$.
- $|x+y| \leq |x| + |y|$.
- $|xy| = |x||y|$.
- $|x-y| = |y-x|$.
- $|x-y| \leq |x-z| + |z-y|$.
- $|x|^2 = x^2$.

8.2.3 Inequalities with absolute value

The fundamental inequalities are given by

$$|x| > a, \quad |x| \geq a, \quad |x| < a, \quad |x| \leq a,$$

where a is any real number. If a is negative the firsts two inequalities are always satisfied. On the contrary, the third and the fourth inequality are never satisfied if $a < 0$ (see the first of the properties itemized above). Instead, the following properties hold if $a \geq 0$:

- $|x| > a \Leftrightarrow x < -a \vee x > a$;
- $|x| \geq a \Leftrightarrow x \leq -a \vee x \geq a$;
- $|x| < a \Leftrightarrow -a < x < a$;
- $|x| \leq a \Leftrightarrow -a \leq x \leq a$.

Example 8.5. $|x| > 0 \Rightarrow x < 0 \vee x > 0$, or, equivalently, $x \neq 0$.

Example 8.6. $|x| \geq 0 \Rightarrow x \leq 0 \vee x \geq 0$, or, equivalently, all the $x \in \mathbb{R}$ (note that it coincides with the first property of the absolute value function).

Example 8.7. $|x| \leq 3 \Rightarrow -3 \leq x \leq 3$.

The graphical representation will be very useful in order to solve inequalities involving the absolute value. In particular, we have the following example.

Example 8.8. $|x| > 2$.

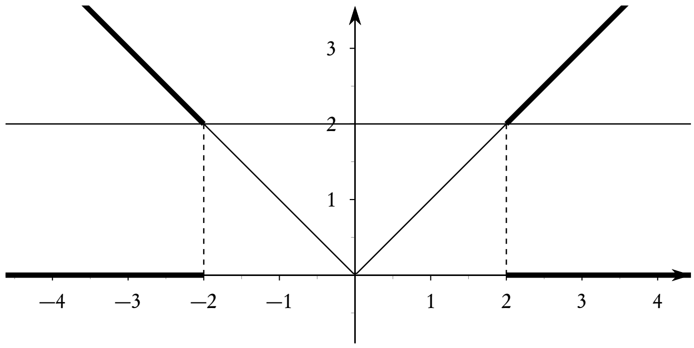


Figure 8.2 *The inequality* $|x| > 2$

Generally, when working with any inequality involving absolute values, one has to examine two cases (think about the definition of absolute value). The following examples will explain the method of resolution.

Example 8.9. $|x| + 2x - 1 > 0$. We need to distinguish two cases.

1. $\begin{cases} x < 0 \\ -x + 2x - 1 > 0 \end{cases} \Rightarrow \begin{cases} x < 0 \\ x > 1 \end{cases}$;
2. $\begin{cases} x \geq 0 \\ x + 2x - 1 > 0 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x > 1/3 \end{cases}$.

The first system does not admit any solution, whereas the second one is satisfied for $x > 1/3$. The range of solutions of the inequality $|x| + 2x - 1 > 0$ is thus $x \in (1/3, +\infty)$.

Example 8.10. $|x - 1| + 3x - 5 < 0$. Because the term $|x - 1|$ can be equal to $x - 1$ or $-(x - 1) = -x + 1$, we need to distinguish two cases.

1. $\begin{cases} x < 1 \\ -x + 1 + 3x - 5 < 0 \end{cases} \Rightarrow \begin{cases} x < 1 \\ x < 2 \end{cases}$;
2. $\begin{cases} x \geq 1 \\ x - 1 + 3x - 5 < 0 \end{cases} \Rightarrow \begin{cases} x \geq 1 \\ x < 3/2 \end{cases}$.

The first system is satisfied for $x < 1$, the second for $1 \leq x < 3/2$. So, the range of solutions of the initial inequality is $x \in (-\infty, 3/2)$.

8.3 Derived graphs of functions

Given a graph of a function, it is possible to build other graphs by applying simple techniques to the starting graph. In what follows, we will analyse some of these procedures applied to the graph of a generic function as in Figure 8.3.

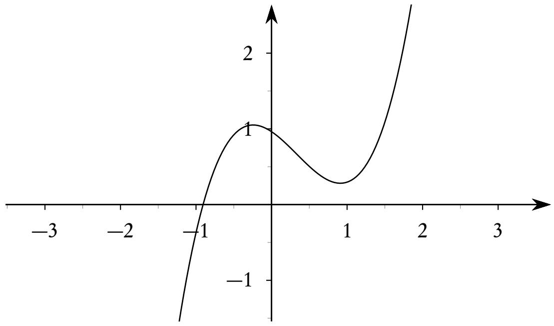
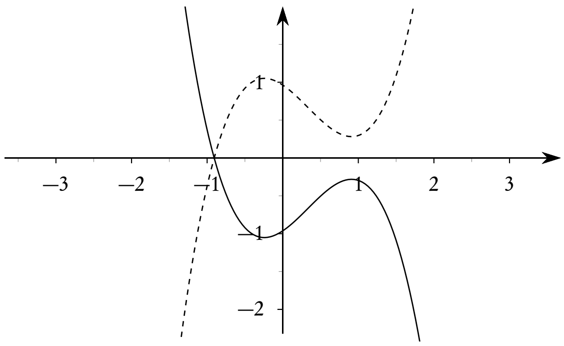


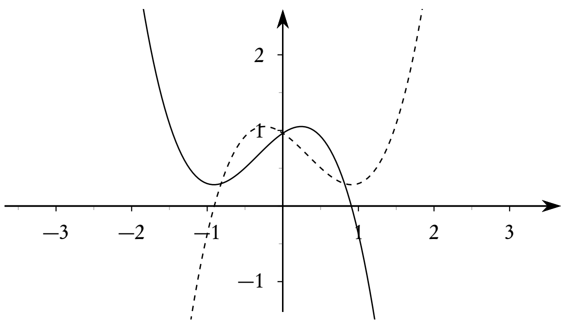
Figure 8.3 *Graph of a generic function* $f(x)$

To get a better understanding, figures below report also, in dotted line, the graph of the initial function of Figure 8.3.

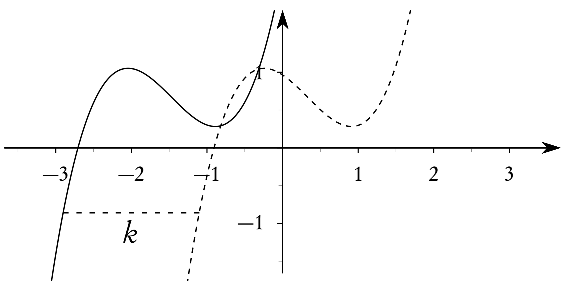
Symmetry with respect to the x-axis: We have to change $f(x)$ in $-f(x)$.



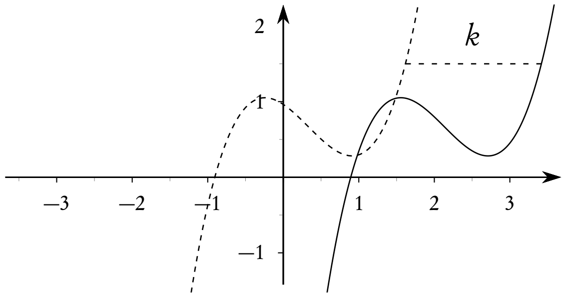
Symmetry with respect to the y-axis: We have to change $f(x)$ in $f(-x)$.



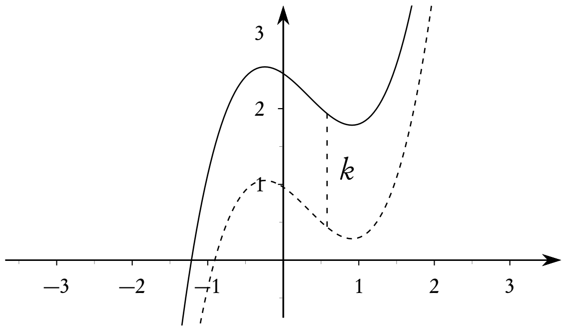
Horizontal translation to the left of k (> 0) unities: We have to change $f(x)$ in $f(x+k)$.



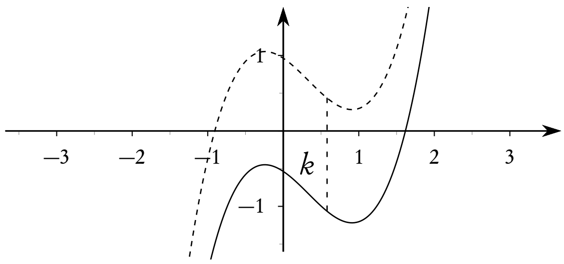
Horizontal translation to the right of k (>0) unities: We have to change $f(x)$ in $f(x-k)$.



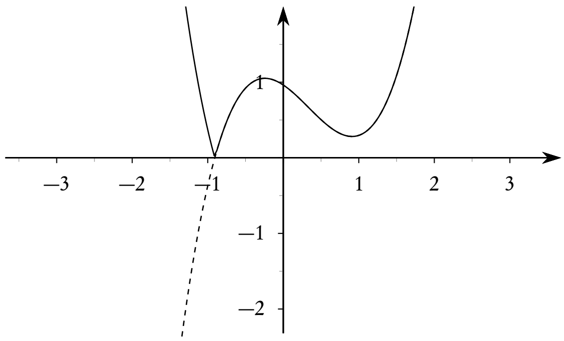
Upward translation of k (>0) unities: We have to change $f(x)$ in $f(x)+k$.



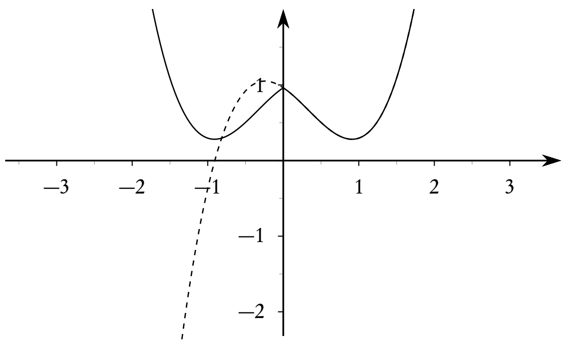
Downward translation of k (>0) unities: We have to change $f(x)$ in $f(x)-k$.



In order to obtain the graph of $|f(x)|$, we have to maintain unchanged the part of the graph above the x-axis and to reflect with respect to the x-axis the part of the graph below the horizontal axis.



In order to obtain the graph of $f(|x|)$, the part of the graph on the right of the y-axis is unchanged. The graph of $f(|x|)$ on the left of the y-axis, instead, is the symmetry of the right side of the graph of $f(x)$ with respect to the y-axis.



It is also possible to combine the previous techniques in order to obtain other graphs.

Example 8.11. Draw the graph of the function $f(x) = -\sqrt{-x} + 1$. We start from the graph of the function $g(x) = \sqrt{x}$ as in the second figure at Page 55. Then, we draw the graph of $\sqrt{-x}$ (*symmetry with respect to the y-axis*) and, subsequently, of $-\sqrt{-x}$ (*symmetry with respect to the x-axis*). Finally, we draw the graph of $-\sqrt{-x} + 1$ (*upward translation of k = 1 unity*). Figures 8.4, 8.5 and 8.6 report the just-described steps. Obviously, the graph of $f(x) = -\sqrt{-x} + 1$ can be also obtained by applying the previous techniques in different order (note that $f(x) = -\sqrt{-x} + 1 = -(\sqrt{-x} - 1)$.

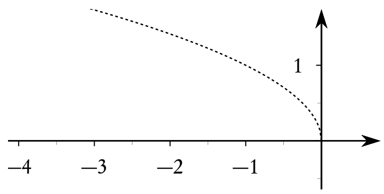


Figure 8.4 *Graph of the function* $\sqrt{-x}$

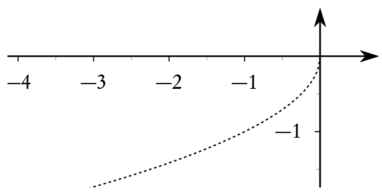


Figure 8.5 *Graph of the function* $-\sqrt{-x}$

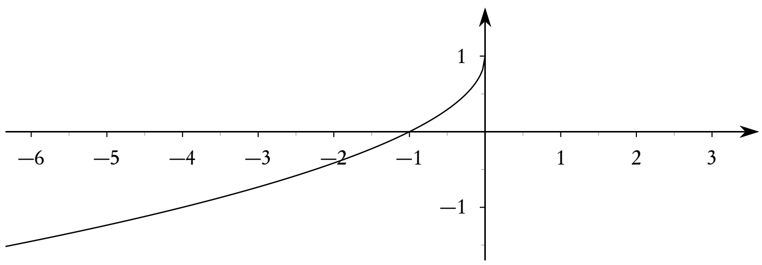


Figure 8.6 *Graph of the function* $-\sqrt{-x} + 1$

Example 8.12. Draw the graph of $f(x) = \sqrt{-x-1}$. In order to use the graphs previously drawn, it is essential to rewrite $f(x) = \sqrt{-x-1}$ as $f(x) = \sqrt{-(x+1)}$. So, in this case the sequence will be: i) draw $\sqrt{x}$, then ii) $\sqrt{-x}$ (see Figure 8.4) and, finally, iii) $\sqrt{-(x+1)}$ (see Figure 8.7).

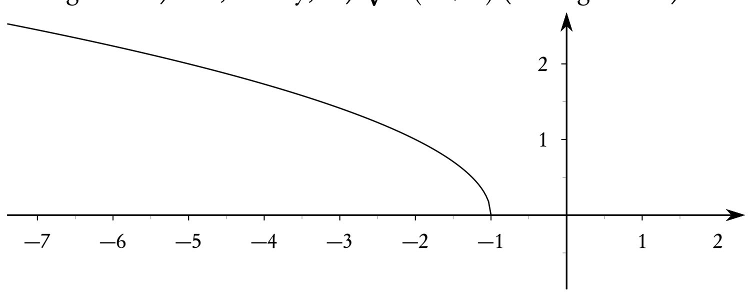


Figure 8.7 *Graph of the function* $\sqrt{-x-1}$

Example 8.13. Draw the graph of $f(x) = \ln(|x - 1|)$. We start from the graph of $\ln x$. Then, we draw the graph of $\ln|x|$ (*the graph on the left of the y-axis is the symmetric of the graph on the right of the vertical* *axis*) and, finally, of $\ln(|x-1|)$ (*horizontal translation to the right of $k=1$ unity*). Figures 8.8 and 8.9 report the just-described steps.

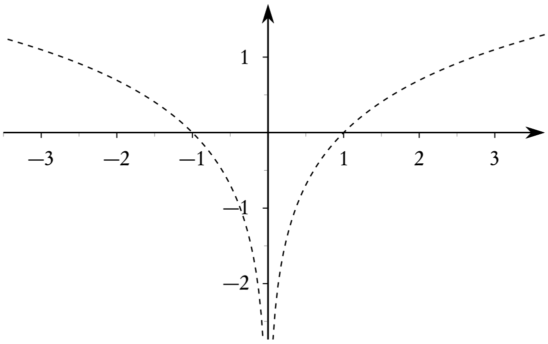


Figure 8.8 *Graph of the function* $\ln|x|$

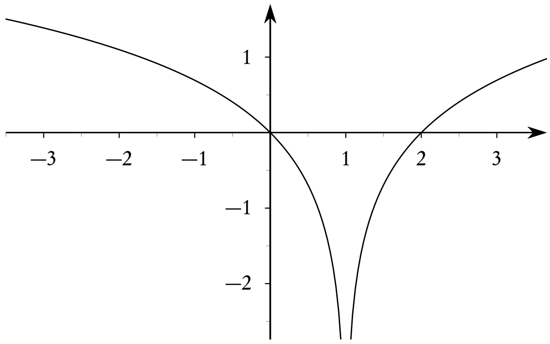


Figure 8.9 *Graph of the function* $\ln|x-1|$

Example 8.14. Draw the graph of $f(x)=\dfrac{1}{|x|-1}$. The sequence of steps required is: i) draw $\dfrac{1}{x}$, then ii) $\dfrac{1}{x-1}$ (*horizontal translation to the right of $k=1$ unity*) and, finally iii) $\dfrac{1}{|x|-1}$ (*the graph on the left of the y-axis is the symmetric of the graph on the right of the vertical axis*). Figures 8.10 and 8.11 report the last two steps. We also report the lines of equation $x=1$ and $x=-1$. In particular, these lines are named *asymptotes*.

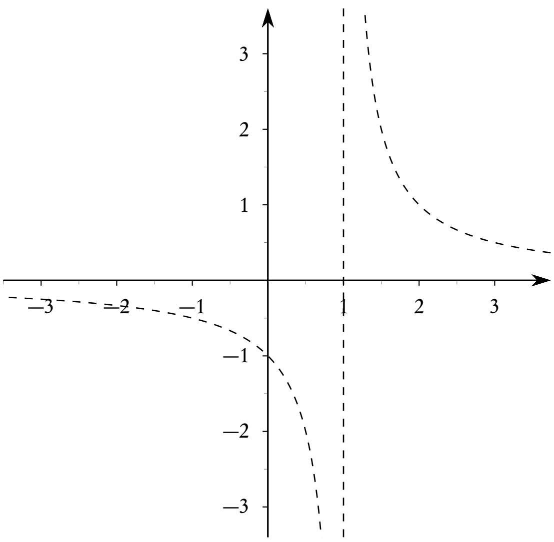


Figure 8.10 *Graph of the function* $1/(x-1)$

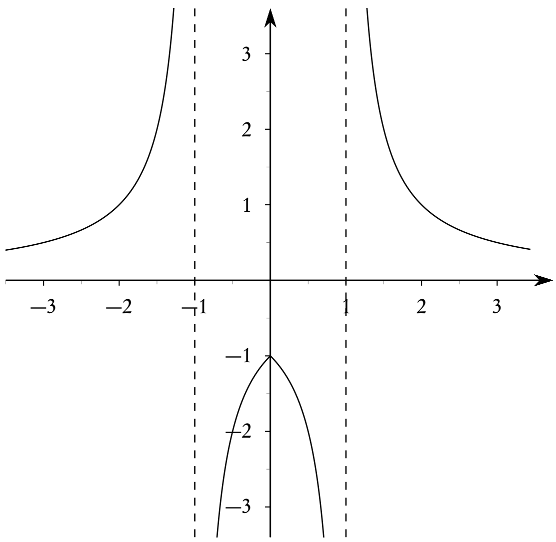


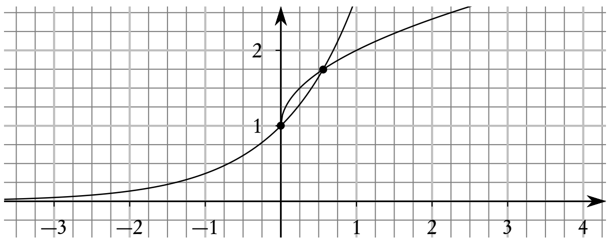
Figure 8.11 *Graph of the function* $1/(|x|-1)$

We note that it is not so difficult to deduce the domain and the image set from the graph of a function. Graphical representation of a function will also help to solve, graphically, a two-variable system of equalities (see Paragraph 3.2 of chapter 3). We note that, in this case, the precise numerical value of the solutions can be determined only approximatively.

Example 8.15. Solve the system

$$\begin{cases} y = \sqrt{x} + 1 \\ y = e^x \end{cases} .$$

First we need to draw the graph of $y = \sqrt{x} + 1$ and of $y = e^x$, then the number and the positions of their intersections are easily deduced.



In particular, there are two intersections. The first one at the point $(0, 1)$, the second at $\approx (0.5, 1.8)$.

The same techniques can be applied in order to solve systems of inequalities more complex than those analysed in the Paragraph 5.3.2 of chapter 5.

8.4 Exercises

Exercice 8.1. *Determine the domain of the following functions and then draw their graph.*

1. 2^{x-1};
2. $-\ln(1-x)$;
3. $\ln(x+1)+2$;
4. $-\ln(-x)-1$;
5. $e^{1-x}-1$;
6. $-(3^x+1)$;
7. $-(\ln(-x+1)-1)$;
8. $\ln(x-2)+3$;
9. $-\ln(-x)+1$;
10. -2^{-x};
11. -2^{-x};
12. -3^{x-1};
13. e^{-x-1};
14. $\sqrt{x-1}+1$;
15. $-\sqrt{-x}+1$;
16. $\sqrt{1-x}+2$;
17. $-(\sqrt{x+2}+1)$;

18. $-x^2+2$;
19. $-(x-2)^2+3$;
20. $(1-x)^3+2$;
21. $-(1-x)^2+2$;
22. $-(x+1)^2-2$;
23. $-x^3+3$.

Exercice 8.2. *Determine the number of solutions of the following systems of equations.*

1. $\begin{cases} y = x^3+1 \\ y = 2-(1-x)^2 \end{cases}$;
2. $\begin{cases} y = 1+(2-x)^3 \\ y = \sqrt{x+1}+2 \end{cases}$;
3. $\begin{cases} y = \ln(x-1)+2 \\ y = 1-\sqrt{x} \end{cases}$;
4. $\begin{cases} y = \mathrm{e}^x \\ y = -\sqrt{x+2}+3 \end{cases}$;
5. $\begin{cases} y = \mathrm{e}^{-x-1} \\ y = \sqrt{x}+3 \end{cases}$;
6. $\begin{cases} y = \ln(x+1) \\ y = \sqrt{x} \end{cases}$;
7. $\begin{cases} y = -\ln(x+1) \\ y = -\dfrac{1}{10}\,x \end{cases}$;
8. $\begin{cases} y = \mathrm{e}^{-x}+3 \\ y = \ln(x-10) \end{cases}$.

Exercice 8.3. *Solve the following systems of inequalities.*

1. $\begin{cases} y+3^{-x}-2>0 \\ x^2+y^2<4 \end{cases}$;
2. $\begin{cases} y-\ln(1-x)+1>0 \\ x^2+y^2\geq 1 \end{cases}$.

Exercice 8.4. *Solve the following inequalities.*

1. $|x|<2$;
2. $|x+1|>3$;
3. $|x^2-1|<x$;
4. $|\mathrm{e}^x-1|>2$;

5. $|\log_2(x) - 1| < 2$;
6. $|x - 1| < 1$.

Exercice 8.5. *Given the function* $f(x) = -\sqrt{-x+1}$, *determine first the image set, then the image of the following sets*

1. $A = [-3, 2]$;
2. $B =]-2, -1[$;
3. $C = [0, 1]$.

9 Sets and Functions: something more

9.1 Bounded and unbounded sets of real numbers

Attention: In this paragraph all the sets are subsets of the set $\mathbb{R}$ of the real numbers.

Definition 9.1. *Let $A \subseteq \mathbb{R}$ be a set. A* majorant *of A is a real number x such that*

$$x \geq a, \ \forall a \in A. \tag{9.1}$$

A real number y is said minorant *of A if*

$$y \leq a, \ \forall a \in A. \tag{9.2}$$

Example 9.1. Let A be the set $A =]-2, 8]$. Then -5, $-\pi$, -2 are minorants; 8, 10, $\sqrt{89}$ are majorants of A

Example 9.2. Let A be the set $A = \mathbb{N}$. There are no majorants of A, while all the real numbers less or equal than zero are minorants of A.

Example 9.3. Let A be the set $A = \mathbb{Z}$. There are no majorants and minorants of A.

Definition 9.2. *Let $A \subseteq \mathbb{R}$ be a set. A is said to be* bounded from above *if it cannot extend indefinitely rightwards on the real line, i.e., if there exists at least a majorant of A. Analogously, A is said to be* bounded from below *if it cannot extend indefinitely leftwards on the real line, i.e., if it there exists at least a minorant of A. A is said to be* bounded *if it is simultaneously bounded from above and from below.*

Example 9.4. The set $\mathbb{N}$ is bounded from below but it is not bounded from above.

Example 9.5. The set $\mathbb{Z}$ is not bounded.

Example 9.6. The set $A =]2, 6[$ is bounded.

Definition 9.3. *A real number M is said to be the* maximum *(respectively, the* minimum*) of a set $A \subseteq \mathbb{R}$ if*

- *$M \in A$ and*
- *for every $a \in A$ one has $a \leq M$ (respectively, $a \geq M$).*

The maximum (respectively, the minimum of a set) is unique.

Example 9.7. Consider the interval $A = [0, 1]$. The maximum of A is 1, while the minimum is 0. Hereafter, we shall use the following notation for the maximum and the minimum of a set A: $\max(A)$ and $\min(A)$. In particular, $1 = \max(A)$ and $0 = \min(A)$.

Example 9.8. Let A be the set $A =]0, 1[$. A has no maximum and no minimum.

Example 9.9. Let A be the set $A =]0, 1]$. Then $1 = \max(A)$, whereas A has not minimum.

Example 9.10. $\min(\mathbb{N}) = 0$, while $\mathbb{N}$ has not maximum.

Example 9.11. $\mathbb{Z}$ has no maximum and no minimum.

By definition, a set which is unbounded from above cannot have maximum and a set which is unbounded from below cannot have minimum. At the same time, it may happen that bounded sets have neither the maximum and the minimum. In particular, we give the following definition.

Definition 9.4. *Let $A \subseteq \mathbb{R}$ a set bounded from above. The minimum of the majorants of A is named* supremum *of A and it is denoted by* $\sup(A)$. *Let now A be a set bounded from below. The maximum of the minorants of A is named* infimum *of A and it is denoted by* $\inf(A)$. *If A is an unbounded set form above, then, by definition,* $\sup(A) = +\infty$*; if A is an unbounded set from below, then, by definition,* $\inf(A) = -\infty$.

Each subset of $\mathbb{R}$ has always a supremum (eventually $+\infty$) and an infimum (eventually $-\infty$). In order to distinguish between bounded and unbounded sets, we name *finite* a *supremum* or *infimum* of bounded sets.

Example 9.12. $+\infty = \sup(\mathbb{N})$, $0 = \inf(\mathbb{N}) = \min(\mathbb{N})$.

Example 9.13. $+\infty = \sup(\mathbb{Z})$, $-\infty = \inf(\mathbb{Z})$.

Example 9.14. $1 = \sup(]0,1[)$, $0 = \inf(]0,1[)$.

Example 9.15. $1 = \sup([0,1]) = \max([0,1])$, $0 = \inf([0,1]) = \min([0,1])$.

If a set has a maximum then the maximum is also a supremum of the set; similarly, if a set has minimum then the minimum is also an infimum of the set. In particular, for a supremum and infimum of a set the following theorem holds.

Theorem 9.5. *If $A \subseteq \mathbb{R}$ is a bounded set from above, then* $\sup(A)$ *satisfies the following properties*

1. $\sup(A) \geq a$, $\forall a \in A$;
2. $\forall \varepsilon > 0$, $\exists a \in A$, $a > \sup(A) - \varepsilon$.

If $A \subseteq \mathbb{R}$ is a bounded set from below, then $\inf(A)$ *satisfies the following properties*

1. $\inf(A) \leq a$, $\forall a \in A$;
2. $\forall \varepsilon > 0$, $\exists a \in A$, $a < \inf(A) + \varepsilon$.

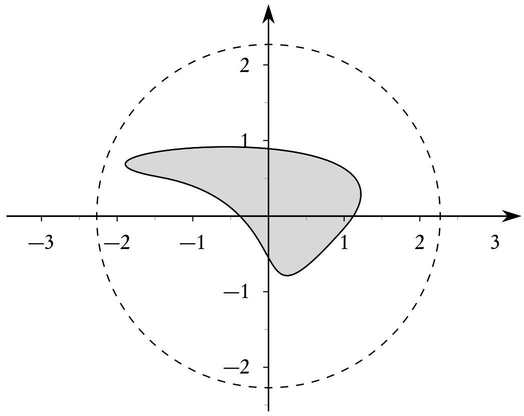
9.2 Bounded and unbounded sets of the plane

The definition of bounded and unbounded set for a set $A \subseteq \mathbb{R}^2$ is different with respect to the definition of bounded and unbounded set for a set $A \subseteq \mathbb{R}$. This is because on the real line there exists an order.

Definition 9.6. *Let $A \subseteq \mathbb{R}^2$ be a subset of the plane. A is said to be* bounded *if it is included into a circle with center at the origin and radius r. Otherwise, it is said to be* unbounded.

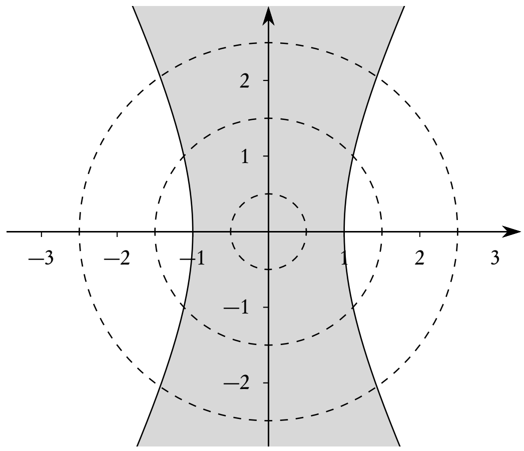
Example 9.16.

A bounded set of the plane and a circle containing it.



Example 9.17.

An unbounded set of the plane: There are no circles containing it.



9.3 Topology

In what follows, we need to use similar concepts for subsets of the line and of the plane. In order to make the definitions uniform, it is convenient to give the following

Definition 9.7. *Let P be a point of the line (P is a real number) or of the plane (P is a pair of real numbers). A* ball *of radius ε and center P is the set of all points of distance less than ε from P; A* closed ball *of radius ε and center P is the set of all points of distance less than or equal to ε away from P.*

It is evident that on the line a ball is an interval (open or closed) of radius ε and center c; on the plane a ball is a circle (open if the circumference does not belong to the circle and close otherwise), of radius ε and center at P.

Definition 9.8 (Neighbourhood)**.** *Given a point P in $\mathbb{R}$ or in $\mathbb{R}^2$, any open ball containing P is called* neighbourhood *of P and it is denoted by $I(P)$ or I_P.*

We are particularly interested in neighbourhoods with center at P and radius ε. In particular, the latters are denoted by $I(P,\varepsilon)$.

Definition 9.9 (Interior point). *Let A be a set. A point $P \in A$ is called an* interior point *of A if there exists an open ball (i.e. a neighbourhood) centered at P that lies entirely in A. An interior point belongs to the set A.*

Definition 9.10 (Exterior point). *Let A be a set. A point P is called an* exterior point *of A if there exists a ball (i.e. a neighbourhood) centered at P that lies entirely on the complementary set of A. An exterior point does not belong to the set A.*

Definition 9.11 (Isolated point). *Let A be a set. A point P of A is called an* isolated point *of A if there exists a neighbourhood $I(P)$ of P such that $I(P) \cap A = \{P\}$, i.e., if P is an element of A but there exists a neighbourhood of P which does not contain any other point of A.*

Definition 9.12 (Boundary point). *Let A be a set. A point P is called a* boundary point *of A if for each neighbourhood $I(P)$ of P it happens that $I(P) \cap A \neq \emptyset$ and, contemporaneously, $I(P) \cap \complement A \neq \emptyset$, i.e. if in each neighbourhood of P there exists at least a point of A and a point of the complementary set of A. A boundary point can belong or not belong to the set A.*

Definition 9.13 (Accumulation point). *Let A be a set. A point P is called an* accumulation point *of A if in each neighbourhood $I(P)$ of P there exists an infinite number of points of A, i.e. the set $I(P) \cap A$ contains an infinite number of points. An accumulation point can belong or not belong to the set A.*

We present now some examples.

Example 9.18. Let $A = [0, 2[\cup \{5\}$ be a subset of the line.

- 1 is an interior point, because the neighbourhood $I(1) =]1/2, 3/2[$ is all contained into A. The set of all the interior points is given by $]0, 2[$.
- 7 is an exterior point, because the neighbourhood $I(5) =]6, 8[$ is all contained in the complementary set of A. The set of all the exterior points is given by $]-\infty, 0[\cup]2, 5[\cup[5, +\infty[$.
- 5 is the only isolated point because the neighbourhood $I(5) =]4, 6[$ intersected with A gives only the point 5.
- 0 is a boundary point because each neighbourhood of 0 contains points that do not belong to A and points that belong to A. Similarly, 2 and 5 are boundary points. We note that $\{0, 5\} \in A$, while $2 \notin A$.
- 1 is an accumulation point, because the neighbourhood $I(1) =]1/2, 3/2[$ contains an infinite number of points of A. Similarly, also 2 is an accumulation point. The set of all the accumulation points is given by $[0, 2]$.

We item now the following properties (*suggestion: try to prove them as an exercise*).

- If P is an interior point than it is also an accumulation point;
- If P is an interior point than it cannot be an isolated point or a boundary point;
- If P is an isolated point than it is always a boundary point;
- If P is an isolated point than it cannot be an accumulation point.

Example 9.19. In this example, the set A is given by union of the line r of equation $x = 2$, the point $P = (1, 1)$ and the circle with center at the origin and radius 1. In particular, the semi-circumference contained on the half-plane $y \geq 0$ belongs to the circle.
The proof of the statements below is left as an exercise.

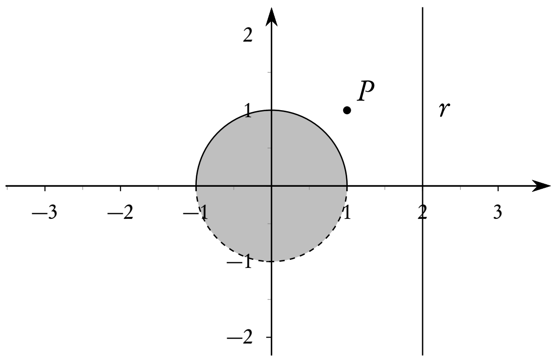


Figure 9.1 *A set of the plane*

- The set of all the interior points is given by all the points which are inside the circle with center at the origin and radius 1.
- The set of all the exterior points is given by all the points which are outside the closed circle with center at the origin and radius 1 with the exclusion of the point P and the points of the line r.
- P is the only isolated point.
- The set of all the boundary points is given by all the points of the *circumference* with center at the origin and radius 1, the point P and the points of the line r.
- The set of all the accumulation points is given by the line r and by the circle with center at the origin and radius 1.
- The set A is an unbounded set of the plane.

Definition 9.14 (Closed set). *A set A is said to be* closed *if it contains all its accumulation points.*

Definition 9.15 (Open set). *A set A is said to be* open *if its complementary is closed.*

Example 9.20. An open (resp. closed) ball is an open (resp. closed) set. An open (resp. closed) interval is an open (resp. closed) set.

Example 9.21. The empty set (as subset of $\mathbb{R}$ or $\mathbb{R}^2$) is both open and closed. Analogously, $\mathbb{R}$ (on the line) and $\mathbb{R}^2$ (on the plane) is both open and closed. Precisely, these are the only sets both open and closed.

Example 9.22. An interval as $[a,b[$ or $]a,b]$ is neither open nor closed.

Example 9.23. The subset of the plane in Figure 9.1 is neither open nor closed.

Example 9.24. The sets $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{Z} \subset \mathbb{R}$ are closed.

We item now some properties whose proof is left as an exercise.

- A set is an open set if and only if all its points are interior points.
- A set is a closed set if and only if it contains all its boundary points.
- If a set has at least an isolated point then it cannot be open.
- If a set has *only* isolated points then it is a closed set.
- Let A and B be two closed sets. Then $A \cup B$ and $A \cap B$ are closed.

– Let A and B be two open sets. Then $A \cup B$ and $A \cap B$ are open. However, if we consider an infinite number of intersections or unions of bounded sets some surprises will arise. For instance, the sets

$$]-1,1[, \left]-\frac{1}{2},\frac{1}{2}\right[, \left]-\frac{1}{3},\frac{1}{3}\right[, \left]-\frac{1}{4},\frac{1}{4}\right[, \dots,$$

are open sets but their intersection is given only by the point 0, which is a closed set.

9.4 Connected sets. Convex sets

Definition 9.16 (Connected set)**.** *A set A (in $\mathbb{R}$ or in $\mathbb{R}^2$) is said to be* arc-wise connected *if any two of its points P and Q are joined by an arc that lies entirely in A*[1]

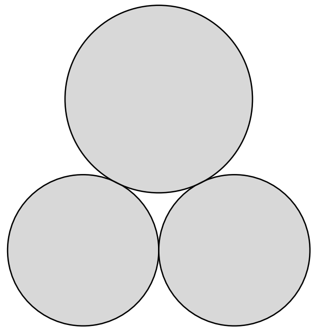
In $\mathbb{R}$ the only connected sets are the intervals. In $\mathbb{R}^2$ an example of connected set is the open ball.

Definition 9.17 (Convex set)**.** *Let A be set (in $\mathbb{R}$ or in $\mathbb{R}^2$). A is said to be* convex *if for any two of its points P and Q, every point on the straight line segment that joins P and Q is also within the set A.*

It is evident that convex sets are also connected. However, on the plane the vice-versa is not always true. In $\mathbb{R}$, instead, the two concepts are equivalent: The only sets that are both convex and connected are the intervals.

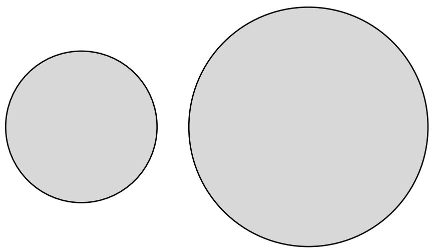
Example 9.25.

Example of a connected but not-convex set.



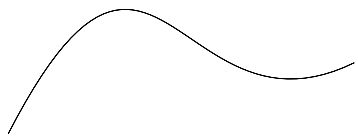
Example 9.26.

Example of a not connected (and thus not convex).



Example 9.27.

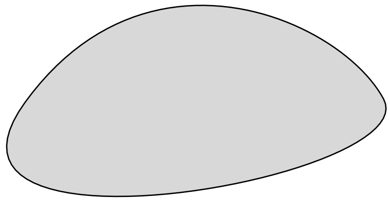
Example of a connected but non-convex set.



[1]Actually, there exits a more complicated definition of connectedness but it is beyond the scopes of this crash introduction.

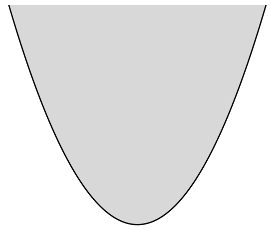
Example 9.28.

Example of a set which is convex and connected.



Example 9.29.

Example of a set which is convex and connected.



9.5 Operations on functions

In this section we suppose that the domain of the functions is a subset A of $\mathbb{R}$ or $\mathbb{R}^2$, whereas the co-domain is always the set $\mathbb{R}$, i.e., we will work with one- or two-variable real functions. It is always possible to sum, subtract or multiply two functions f and g; besides, it is possible to divide two functions only if the second one is always different from zero.

Example 9.30. Let $f(x) = |x|$ and $g(x) = x^2 + 1$ be two real functions with domain $\mathbb{R}$. Their sum is $|x| + x^2 + 1$, the difference is $|x| - x^2 - 1$, the product is $|x|(x^2 + 1)$ and, finally, the ratio is equal to $|x|/(x^2 + 1)$.

Example 9.31. If $f(x) = e^x$ and $g(x) = x^2$, it is always possible to compute the sum and the product of these functions; however, to compute the ratio f/g, it is necessary to restrict the domain of the function g to $\mathbb{R} \setminus \{0\}$.

Under some conditions, it is possible to *compose* two functions f and g. Precisely, the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ can be *composed* to yield a function, which maps the point x in the domain of the function f to the point $g(f(x))$ in the image of the function g. For example, if $f(x) = x^2$ and $g(x) = e^x$, then $g(f(x)) = e^{x^2}$.

Note 9.1. In order to compose two functions f and g, it is important that the image set of the function f is a subset of the domain of the function g. For instance, if $f(x) = x^2 - 1$ and $g(x) = \sqrt{x}$, then we must impose the condition $x^2 - 1 \geq 0$.

9.6 Elementary functions and piecewise defined functions

In mathematics, an *elementary function* is a one- or two-variable function defined as the composition of a finite number of arithmetic operations (+, −, ×, ÷), exponentials, logarithms, constants, and solutions of algebraic equations. The most common way used to define non-elementary functions is to construct these functions "piece by piece". Broadly speaking, we have to "combine" (the graphs of) two functions defined on different subsets of $\mathbb{R}$ (or $\mathbb{R}^2$).

Example 9.32. Let $f(x)=\sqrt{x}$ and $g(x)=\sqrt{-x+1}$ be two functions. The domain of the first function is the range $x \geq 0$, while the domain of the second one is $x \leq 1$. Figures 9.2 and 9.3 report the graphs of f and g, respectively.

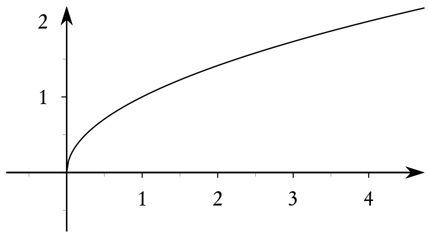


Figure 9.2 *Graph of the function* $\sqrt{x}$

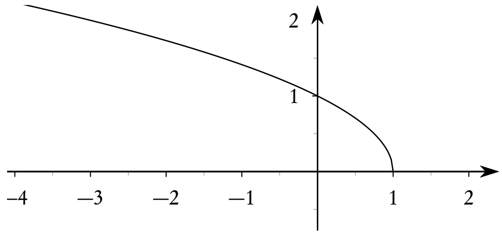
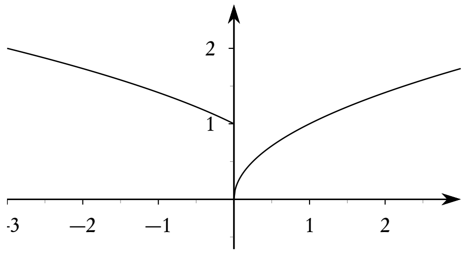


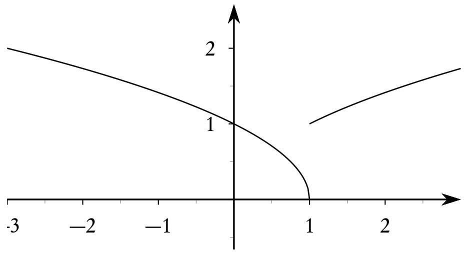
Figure 9.3 *Graph of the function* $\sqrt{-x-1}$

In what follows, we show how to construct other functions starting from f and g.

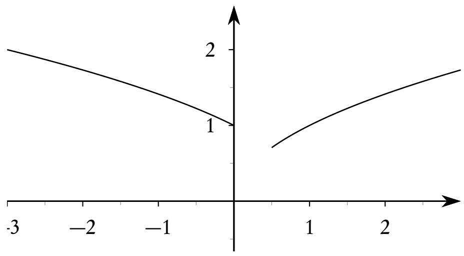
$$g(x)=\begin{cases}\sqrt{-x+1}, & \text{if} \quad x<0\\ \sqrt{x}, & \text{if} \quad x\geq 0\end{cases}.$$



$$h(x)=\begin{cases}\sqrt{-x+1}, & \text{if} \quad x<1\\ \sqrt{x}, & \text{if} \quad x\geq 1\end{cases}.$$



$$k(x)=\begin{cases}\sqrt{-x+1}, & \text{if} \quad x<0\\ \sqrt{x}, & \text{if} \quad x\geq 1\end{cases}.$$



9.7 Domain of elementary functions

We have seen that in order to define a function (see 2.6, Page 11) it is necessary to specify i) its domain, ii) its co-domain, and iii) a law associating to each point on the domain a single point into the co-domain. The domain of an elementary function is the largest subset in which the operations to be executed on the variables make sense. The following example will clarify this concept.

Example 9.33. Determine the domain of $f(x) = \sqrt{x-1} + \ln(2-x)$. To do so, we need to consider the following system of equations

$$\begin{cases} x-1 \geq 0 \\ 2-x > 0 \end{cases},$$

Indeed, the domain of a radical function is given by the set of real numbers greater or equal than zero, while the domain of a logarithmic function is given by the set of positive real numbers. So, the domain of f is $1 \leq x < 2$.

Example 9.34. Determine the domain of $f(x,y) = \sqrt{x^2+y^2-1}$. (*Exercise*)

9.8 Increasing and decreasing functions

Attention: The concepts of increasing and decreasing functions are meaningful *only* for *one-variable functions*.

Definition 9.18 (Increasing and decreasing functions). *Let $f : A \to \mathbb{R}$ be a one-variable function. If for each pair of points x_1 and x_2 in A the condition $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$, then f is said to be* weakly increasing *(if, instead, $x_1 < x_2$ implies $f(x_1) < f(x_2)$ then f is said to be* strictly increasing*). On the other hand, if for each pair of points x_1 and x_2 in A the condition $x_1 < x_2$ implies $f(x_1) \geq f(x_2)$, f is said to be* weakly decreasing *(if, instead, $x_1 < x_2$ implies $f(x_1) > f(x_2)$ then f is said to be* strictly decreasing*).*

The functions e^x, $\ln x$, $\sqrt{x}$ are strictly increasing functions; the function x^2 is neither increasing nor decreasing. In particular, if a function is neither increasing nor decreasing, it is possible that it is *piecewise increasing* or *piecewise decreasing*. For instance, the function x^2 is decreasing for $x < 0$ and increasing for $x > 0$; the function $1/x$ is increasing for $x < 0$ and $x > 0$. In the latter case, however, we need to pay attention on the domain.

9.9 Bounded and unbounded functions, maximum and minimum

The concepts of bounded and unbounded functions are linked to the image set of a function.

Definition 9.19 (Bounded and unbounded functions). *If the whole graph of a function f, defined on a domain A, lies under some horizontal line of equation $y = K$, the function is said to be* bounded from above. *It means that*

$$f(x) \leq K,$$

for each $x \in A$. Similarly, f is said to be bounded from below *if the graph has no points under some line of equation $y = H$, i.e. if*

$$f(x) \geq H,$$

for each $x \in A$. A function which is bounded both from below and above is said to be bounded. *A function whose image set is unbounded is said to be* unbounded.

For instance, the trigonometric functions $\sin x$ and $\cos x$ are bounded, the function $1/x$ is unbounded, the function e^x is *bounded from below* and the function $-x^2$ is *bounded from below*.

We have seen that some points on the graph of f correspond to a maximum or minimum height. Now we give a precise definition for these type of points.

Definition 9.20 (Maximum and minimum of a function)**.** *Let f be a function with domain A. A real number M is called the (global)* maximum *of f in A and $P \in A$ is called the (global)* maximum point*, if, for each $x \in A$,*

$$M = f(P) \geq f(x).$$

Similarly, a real number m is called the (global) minimum *of f in A and $P \in A$ is called the (global)* minimum point*, if, for each $x \in A$,*

$$m = f(P) \leq f(x).$$

A maximum (or minimum) is said to be strict if the equal sign holds only for $x = P$. If the maximum and minimum of f do exist, than they are unique.

Finding the global maximum and minimum of a function is one of the most important application in *optimization* problems.

For instance, the function e^x has neither max nor min, its inf is 0 and its sup is $+\infty$. The function $\ln x$ has neither max nor min, its inf is $-\infty$ and its sup is $+\infty$. The function $\sin x$ has 1 as max and -1 as min. The function x^2 has 0 as min, while it has not max; its sup is $+\infty$.

Definition 9.21 (Relative maximum and minimum)**.** *Let f be a function with domain A. A point $P \in A$ is said to be a* relative maximum point *if $\exists I(P)$ such that, for each point $Q \in I(P)$,*

$$f(Q) \leq f(P).$$

The point P is said to be a relative minimum point *if*

$$f(Q) \geq f(P).$$

Relative maximum and minimum points may be many, and also infinitely many.

Example 9.35. The function represented in Figure 9.4 has x_1 and x_3 as relative maximum points, x_2 and x_4 as relative minimum points; it has not global minimum and maximum.

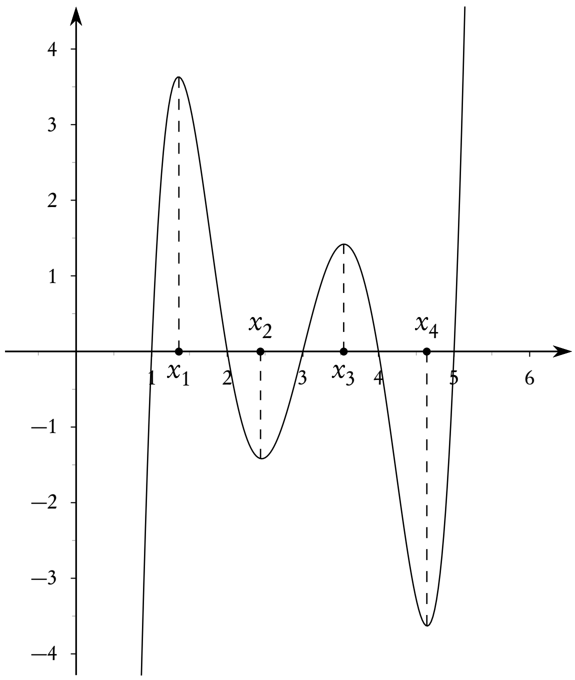


Figure 9.4 *Example of a function with two maximum and two minimum relative points*

Example 9.36. Figure 9.5 represents a function with an infinite number of relative maximum and minimum point.

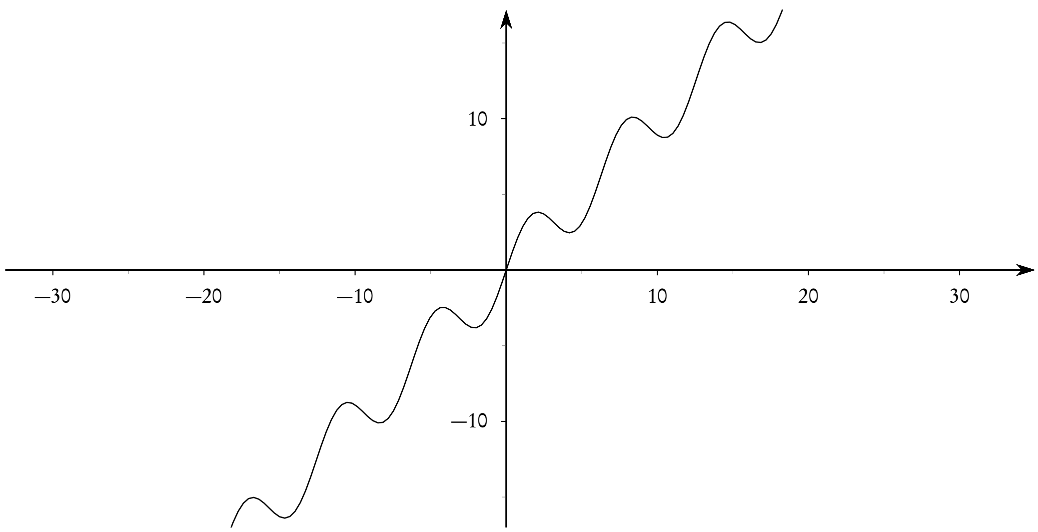


Figure 9.5 *Example of a function with an infinite number of maximum and minimum relative points*

Example 9.37. In the case of two-variable functions something similar happens, but plotting the graph is much more difficult. Figure 9.6 shows an example.

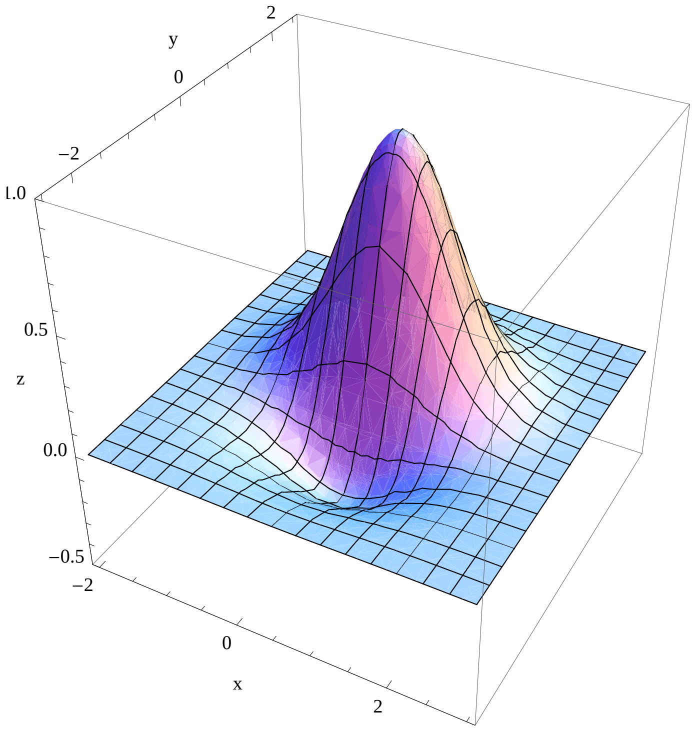


Figure 9.6 *Maximum and minimum points of a two-variable function*

9.10 Injective, surjective and bijective functions

Definition 9.22. *Let $f : A \to B$ be a function. f is said to be i)* injective *if it never maps two distinct elements P_1 and P_2 of its domain to the same element of its co-domain, ii)* surjective *if every element of its co-domain has a corresponding element in its domain , iii)* bijective *or a* bijection *if it is both* injective *and* surjective.

Example 9.38. The functions e^x, $\ln x$, x^3, $\sqrt{x}$ are all injective functions.

Example 9.39. The functions $\ln x$ and x^3, with co-domain $\mathbb{R}$, are surjective functions. The functions e^x and $\sqrt{x}$ are surjective functions only if we restrict their co-domain to $y > 0$ and $y \geq 0$, respectively.

Example 9.40. The real function $f(x) = x^3$ is both injective and surjective.

Example 9.41. The functiom x^2 is not injective. In particular, $f(1) = f(-1) = 1$.

9.11 Exercises

Exercice 9.1. *Determine the characteristics (open, closed, bounded...) of the each of each set of solutions of Exercise 5.1, Page 40.*

Exercice 9.2. *Determine the characteristics of each set of solutions of Exercise 5.2, Page 41.*

Exercice 9.3. *Determine the characteristics of each set of solutions of Exercise 2.1, Page 17.*

Exercice 9.4. *Determine the characteristics of each set of solutions of Exercise 2.2, Page 17.*

Exercice 9.5. *Discuss the following statements.*

- *It is possible to find an open and a closed set such that their union is an open set.*
- *It is possible to find an open and a closed set such that their intersection is an open set.*
- *It is possible to find two open sets such that their union is an open set.*
- *It is possible to find two closed sets such that their union is a closed set.*

Exercice 9.6. *Discuss the following statements.*

- *If we construct a function piece by piece (two pieces) using bounded functions, we always obtain a bounded function.*
- *If we construct a function piece by piece (two pieces) using a bounded function and an unbounded function, we can obtain a bounded function.*
- *If we construct a function piece by piece (two pieces) using an increasing function and a decreasing function we can obtain an increasing function.*
- *If we construct a function piece by piece (two pieces) using two decreasing functions we can obtain a decreasing function.*

Exercice 9.7. *Determine the characteristics of each function of Exercise 8.1 at Page 63.*

Exercice 9.8. *Determine the domain, along with its characteristics, of the following function.*

1. $f(x) = x + 1$;
2. $f(x) = \dfrac{x}{2-x}$;
3. $f(x) = \sqrt{x+1}$;
4. $f(x) = \sqrt{x} \cdot \dfrac{1}{1+x}$;
5. $f(x) = \sqrt{\dfrac{x}{2-x}}$;
6. $f(x) = \sqrt{(x-1)(1+x)}$;
7. $f(x) = \dfrac{\sqrt{x}}{\sqrt{2x-3}}$;
8. $f(x) = \sqrt{\dfrac{x}{2x-3}}$;
9. $f(x) = \dfrac{\sqrt{x^2-9}}{3-x}$;
10. $f(x) = \sqrt{2x}\sqrt{x+3}$;

11. $f(x) = \sqrt{x+1} - x + \sqrt{2-x}$.

Exercice 9.9. *Determine the characteristics of each set of solutions of Exercise 8.3, Page 64.*

Exercice 9.10. *Determine the domain, along with its characteristics, of the following functions.*

1. $f(x,y) = \dfrac{x}{2-y}$;
2. $f(x,y) = \sqrt{(x+1)(y+1)}$;
3. $f(x,y) = \sqrt{x} \cdot \dfrac{1}{1+y}$;
4. $f(x,y) = \sqrt{\dfrac{y}{1+x}}$;
5. $f(x,y) = \sqrt{(x^2-1)(2-y)}$;
6. $f(x,y) = \sqrt{xy}$;
7. $f(x,y) = \dfrac{\sqrt{x^2-4}}{y+3}$;
8. $f(x,y) = \sqrt{2x}\sqrt{2y-3}$;
9. $f(x,y) = \sqrt{x+y-1} - \sqrt{x-y}$.

List of Symbols

The notations used in these notes are the standard notations of the typesetting system $\LaTeX\,2_\varepsilon$ [(1)].

Notations

$\neg$	"not" (negation sign).
$\vee$	"or", or both (disjunction sign).
$\wedge$	"and", (conjunction sign).
$\Rightarrow$	"implies", if ... then ... (implication sign).
$\Leftrightarrow$	"if and only if" (equivalence sign).
$\mathbb{N}$	The set of natural numbers: $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$.
$\mathbb{Z}$	The set of integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
$\mathbb{Q}$	The set of rational numbers: $\mathbb{Q} = \{ m/n \mid m \in \mathbb{Z},\ n \in \mathbb{N},\ n \neq 0 \}$.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$\mathbb{N}^+, \mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+$	Natural, integer, rational, real numbers greater than 0.
$A, B, \dots$	Sets notation.
$A \subseteq B$	A is a subset of B.
$A \subset B$	A is a proper subset of B.
$B \supseteq A$	B includes A (as subset).
$B \supset A$	B includes A properly.
$A \setminus B$	difference between A and B.
$[a, b]$	$\{ x \in \mathbb{R} \mid a \leq x \leq b \}$.
$]a, b[$	$\{ x \in \mathbb{R} \mid a < x < b \}$.
$]a, b]$	$\{ x \in \mathbb{R} \mid a < x \leq b \}$.
$[a, b[$	$\{ x \in \mathbb{R} \mid a \leq x < b \}$.
$[a, +\infty[$	$\{ x \in \mathbb{R} \mid x \geq a \}$.
$]a, +\infty[$	$\{ x \in \mathbb{R} \mid x > a \}$.
$]-\infty, a]$	$\{ x \in \mathbb{R} \mid x \leq a \}$.
$]-\infty, a[$	$\{ x \in \mathbb{R} \mid x < a \}$.
$f : D \to C,\ x \mapsto f(x)$	Notation for function.
$\exp(x) = e^x$	Exponential function to the base e of x.
$\ln(x)$	Natural logarithm (to the base e) of x.
$\log(x)$	Common logarithm (to the base 10) of x.

[1] $\LaTeX\,2_\varepsilon$ is an open-source professional typesetting system. All the related information can be found on-line both in English (http://www.ctan.org/) and in Italian (http://www.guit.sssup.it/). Some introductory manuals are available on the Luciano Battaia's personal web-page (http://www.batmath.it)

Observations

- In some textbooks the set of natural numbers $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ does not contain the 0 element.
- The set of rational numbers is the set of fractions equipped with an opportune relation. Precisely, this relation makes equal two equivalent fractions.
- We denote sets using capital letters ($A, B, \dots$). In some textbooks sets are denoted using bold capital letters ($\mathbf{A}$, $\mathbf{B}$, ...). The latter choice, however, can be a bit misleading (for instance, also points into space are denoted by bold capital letters).
- Many authors use the symbol $\subset$ to denote generic subsets (proper or improper subsets) and $\subsetneq$, or $\subsetneqq$, to denote improper subsets.
- Many authors use the symbol $A - B$ to indicate the difference between A and B.
- We utilize the notation commonly found in many calculators and software. However, many authors use the symbol $\log(x)$ to indicate the logarithm to the base e and $\mathrm{Log}(x)$, or simply $\log_{10}(x)$, to denote the logarithm to the base 10 of x.

The Greek Alphabet

Because of the frequent use of Greek letters, we report, for completeness, the entire Greek alphabet:

alpha	α	A	nu	ν	N
beta	β	B	csi	ξ	Ξ
gamma	γ	Γ	omicron	o	O
delta	δ	Δ	pi	π	Π
epsilon	ε	E	rho	ϱ	R
zeta	ζ	Z	sigma	σ	Σ
eta	η	H	tau	τ	T
theta	ϑ	Θ	upsilon	υ	Υ
iota	ι	I	phi	φ	Φ
kappa	$\varkappa$	K	chi	χ	X
lambda	λ	Λ	psi	ψ	Ψ
mu	μ	M	omega	ω	Ω

In mathematics, another letter of common use is the Hebrew letter

$$\text{aleph}\qquad \aleph.$$

Index

Precalculus

A Prelude to Calculus with Exercises

Luciano Battaia, Giacomo Bormetti, Giulia Livieri

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This work is meant for the students that have to attend a university Math course to take a degree in Economics.
It may help also for a quick revision of the preliminary Math concepts usually taught in secondary school and assumed to be well known.